

ON THE QUADRATIC TWIST OF ELLIPTIC CURVES WITH FULL 2-TORSION

ZHANGJIE WANG AND SHENXING ZHANG

ABSTRACT. Let $E : y^2 = x(x - a^2)(x + b^2)$ be an elliptic curve with full 2-torsion group, where a and b are coprime integers and $2(a^2 + b^2)$ is a square. Assume that the 2-Selmer group of E has rank two. We characterize all quadratic twists of E with Mordell-Weil rank zero and 2-primary Shafarevich-Tate groups $(\mathbb{Z}/2\mathbb{Z})^2$, under certain conditions. We also obtain a distribution result for these elliptic curves.

1. INTRODUCTION

A positive integer n is called a *congruent number* if and only if it is the area of a right rational triangle. This is equivalently to say, the congruent elliptic curve $y^2 = x^3 - n^2x$ has positive Mordell-Weil rank. The classical descent method provides a way to construct non-congruent numbers, see [Fen96, LT00, OZ14, OZ15]. In [Wan16], the first author used Cassels pairing to characterize all congruent elliptic curves $y^2 = x^3 - n^2x$ with Mordell-Weil rank zero and second minimal 2-primary Shafarevich-Tate group, where all prime divisors of n are congruent to 1 modulo 4.

Fujiwara in [Fuj98] defined the generalized concept, a θ -congruent number by considering rational triangles with an angle θ . For such a triangle,

$$\cos \theta = \frac{s}{r} \in \mathbb{Q}, \quad s, r \in \mathbb{Z}, r > 0, \gcd(r, s) = 1.$$

A positive integer n is called θ -congruent if $n\sqrt{r^2 - s^2}$ is the area of a rational triangle with an angle θ . This is equivalently to say, the elliptic curve

$$y^2 = x(x + (r + s)n)(x - (r - s)n)$$

has positive Mordell-Weil rank. The goal of this paper is to generalize the result in [Wan16] to non- θ -congruent case. More precisely, we will show that for certain n , n is non-congruent with second minimal 2-primary Shafarevich-Tate group, if and only if n is non- θ -congruent with second minimal 2-primary Shafarevich-Tate group, where both of $\sqrt{2} \sin(\theta/2)$ and $\sqrt{2} \cos(\theta/2)$ are rational.

Let (a, b, c) be a primitive triple of positive integers such that $a^2 + b^2 = 2c^2$. By elementary number theory, this is equivalent to say,

$$a = |\alpha^2 - 2\alpha\beta - \beta^2|, \quad b = |\alpha^2 + 2\alpha\beta - \beta^2|, \quad c = \alpha^2 + \beta^2$$

for some coprime integers α, β with different parities. Denote by

$$E : y^2 = x(x - a^2)(x + b^2)$$

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an elliptic curve with full 2-torsion group, and

$$E^{(n)} : y^2 = x(x - a^2n)(x + b^2n)$$

a quadratic twist of E , where n is a positive square-free integer. If

$$\sin\left(\frac{\theta}{2}\right) = \frac{a}{\sqrt{2c}}, \quad \cos\left(\frac{\theta}{2}\right) = \frac{b}{\sqrt{2c}} \quad \text{and} \quad \tan\left(\frac{\theta}{2}\right) = \frac{a}{b},$$

then $\cos \theta = \frac{b^2 - a^2}{b^2 + a^2}$. Therefore, n is non- θ -congruent if and only if $\text{rank}_{\mathbb{Z}} E^{(n)}(\mathbb{Q}) = 0$.

1.1. Rank zero twists. When $n > 1$, denote by $\mathcal{A}$ the ideal class group of $K = \mathbb{Q}(\sqrt{-n})$ and

$$h_{2^m}(n) := \dim_{\mathbb{F}_2} \frac{\mathcal{A}^{2^{m-1}}}{\mathcal{A}^{2^m}}$$

its 2^m -rank for a positive integer m . Denote by $\text{Sel}_2(E^{(n)}/\mathbb{Q})$ the 2-Selmer group of $E^{(n)}$ over $\mathbb{Q}$.

Theorem 1.1 (= [Theorems 4.2](#) and [4.4](#)). *Assume that $\text{Sel}_2(E/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$. Let $n \equiv 1 \pmod{8}$ be a positive square-free integer coprime to abc where each prime factor of n is a quadratic residue modulo every prime factor of abc .*

- (A) *If all prime factors of n are congruent to ± 1 modulo 8, then the following are equivalent:*
 - (1) $\text{rank}_{\mathbb{Z}} E^{(n)}(\mathbb{Q}) = 0$ and $\text{III}(E^{(n)}/\mathbb{Q})[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2$;
 - (2) $h_4(n) = 1$ and $h_8(n) = 0$.
- (B) *If all prime factors of n are congruent to 1 modulo 4, then the following are equivalent:*
 - (1) $\text{rank}_{\mathbb{Z}} E^{(n)}(\mathbb{Q}) = 0$ and $\text{III}(E^{(n)}/\mathbb{Q})[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2$;
 - (2) $h_4(n) = 1$ and $h_8(n) \equiv \frac{d-1}{4} \pmod{2}$.

Here d is the odd part of $d_0 \mid 2n$ such that the ideal class $[(d_0, \sqrt{-n})]$ is the non-trivial element in $\mathcal{A}[2] \cap \mathcal{A}^2$.

Remark 1.2. (1) When $(a, b) = (1, 1), (7, 23), (23, 47), (119, 167), (167, 223), (287, 359)$, we have $\text{Sel}_2(E/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$.

- (2) In [Theorem 1.1 \(B\)](#), if $h_4(n) = 1$, then the non-trivial element in $\mathcal{A}[2] \cap \mathcal{A}^2$ is $[(d_0, \sqrt{-n})]$ for some positive divisor d_0 of $2n$. If d'_0 is another positive divisor of $2n$ such that $[(d_0, \sqrt{-n})] = [(d'_0, \sqrt{-n})]$, then $d_0 d'_0 = n$ or $4n$. See [Section 2.1](#).

We will first show that $E_{\text{tor}}^{(n)}(\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$ in [Section 2.2](#). In [Section 3](#), we will study the local solvability of homogeneous spaces and then express the 2-Selmer group as the kernel of the generalized Mordell matrix $\mathcal{M}_n$. Then we will give the proof of [Theorem 1.1](#) in [Section 4](#). The strategy is similar to [\[Wan16\]](#).

1.2. Distribution. We will also study the distribution of n satisfying the equivalent condition in [Theorem 1.1 \(B\)](#), which generalizes the result in [\[Wan17\]](#). Denote by

- $C_k(x)$ the set of positive square-free integers $n \leq x$ with exactly k prime factors;
- $\mathcal{Q}_k(x)$ the set of $n \in C_k(x)$ coprime to abc such that each prime factor of $n \equiv 1 \pmod{8}$ is a quadratic residue modulo every prime factor of abc and congruent to 1 modulo 4;

- $\mathcal{P}_k(x)$ the set of all $n \in \mathcal{Q}_k(x)$ such that [Theorem 1.1 \(B\) \(2\)](#) holds.

We will use the standard symbols in analytic number theory: “ $\sim$, $\ll$, $O(\cdot)$, $o(\cdot)$, $\text{Li}(x)$ ”, which can be found in [\[IR90\]](#). It’s a classical result, due to Landau in [\[Lan18, Kapitel XIII, § 56\]](#), that

$$\#C_k(x) \sim \frac{x(\log \log x)^{k-1}}{(k-1)! \log x}. \quad (1.1)$$

Theorem 1.3. *Assume that $\text{Sel}_2(E/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$. Then*

$$\#\mathcal{P}_k(x) \sim 2^{-k\ell-k-2} (u_k + (2^{-1} - 2^{-k})u_{k-1}) \cdot \#C_k(x),$$

where ℓ is the number of different prime factors of abc and

$$u_k := \prod_{1 \leq i \leq k/2} (1 - 2^{1-2i}).$$

We will use the method in [\[CO89\]](#) to show the equidistribution property of residue symbols in [Section 5.3](#) and then use this to prove [Theorem 1.3](#) in [Section 6](#).

Remark 1.4. If one wants to study the distribution of n satisfying the equivalent condition in [Theorem 1.1 \(A\)](#), one needs a characterization of $h_8(n)$ in terms of residue symbols similar to [\[JY11\]](#).

1.3. Notations.

- $n = p_1 \cdots p_k$ the prime decomposition of n .
- $abc = q_1^{t_1} \cdots q_\ell^{t_\ell}$ the prime decomposition of abc .
- $\gcd(m_1, \dots, m_t)$ the greatest common divisor of integers $m_1, \dots, m_t$.
- $\text{Sel}'_2(E^{(n)}) = \text{Sel}_2(E^{(n)})/E^{(n)}(\mathbb{Q})[2]$ the pure 2-Selmer group of $E^{(n)}$, see [Eq. \(2.4\)](#).
- D_Λ the homogeneous space associated to a rational triple (d_1, d_2, d_3) , see [Eq. \(2.2\)](#).
- $(\alpha, \beta)_v$ the Hilbert symbol, $\alpha, \beta \in \mathbb{Q}_v^\times$.
- $[\alpha, \beta]_v$ the additive Hilbert symbol, i.e., the image of $(\alpha, \beta)_v$ under the isomorphism $\{\pm 1\} \xrightarrow{\sim} \mathbb{F}_2$.
- $\left(\frac{\alpha}{\beta}\right) = \prod_{p|\beta} (\alpha, \beta)_p$ the Jacobi symbol with $p \mid \beta$ counted with multiplicity, where $\gcd(\alpha, \beta) = 1$ and $\beta > 0$.
- $\left[\frac{\alpha}{\beta}\right]$ the additive Jacobi symbol, i.e., the image of $\left(\frac{\alpha}{\beta}\right)$ under the isomorphism $\{\pm 1\} \xrightarrow{\sim} \mathbb{F}_2$.
- $\mathcal{D}(K)$ the set of positive square-free divisors of $2n$.
- $\mathbf{0} = (0, \dots, 0)^T$ and $\mathbf{1} = (1, \dots, 1)^T$.
- $\mathbf{I}$ the identity matrix and $\mathbf{O}$ the zero matrix.
- $\mathbf{A} = \mathbf{A}_n$ a matrix associated to n , see [Eq. \(3.1\)](#).
- $\mathbf{R}_n$ the Rédei matrix of $K = \mathbb{Q}(\sqrt{-n})$, see [Eq. \(2.1\)](#).
- $\mathbf{D}_u = \text{diag}\left\{\left[\frac{u}{p_1}\right], \dots, \left[\frac{u}{p_k}\right]\right\}$.
- $\mathbf{b}_u = \mathbf{D}_u \mathbf{1} = \left(\left[\frac{u}{p_1}\right], \dots, \left[\frac{u}{p_k}\right]\right)$.
- $\mathbf{M}_n$ the Monsky matrix associated to n , see [Eq. \(3.2\)](#).
- $\mathcal{M}_n$ the generalized Monsky matrix associated to $E^{(n)}$, see [Eq. \(3.3\)](#).
- $I = \sqrt{-1}$.

- $\mathcal{P}$ the set of primary primes of $\mathbb{Z}[I]$ with positive imaginary part.
- $\left(\frac{\alpha}{\lambda}\right)_2$ the quadratic residue symbol over $\mathbb{Z}[I]$, see Eq. (5.1).
- $\left(\frac{\alpha}{\lambda}\right)_4$ the quartic residue symbol over $\mathbb{Z}[I]$, see Eq. (5.2).
- $\left(\frac{a}{d}\right)_4 := \left(\frac{a}{\lambda}\right)_4$ the rational quartic residue symbol, see Eq. (5.3).
- $\Lambda(\mathfrak{a})$ the von Mangoldt function, see Eq. (5.4).
- χ_0 the trivial character modulo a given integral ideal, see Section 5.2.
- $\psi(x, \chi) = \sum_{\mathfrak{N}\mathfrak{a} \leq x} \chi(\mathfrak{a})\Lambda(\mathfrak{a})$, see Eq. (5.5).
- $C_k(x, \alpha, \mathbf{B})$, $C'_k(x, \alpha, \mathbf{B})$, $T_k(x)$, $T'_k(x)$ sets associated to x , α , $\mathbf{B}$, see Section 5.3.
- $\binom{k}{2} = k(k-1)/2$ the binomial coefficient.

2. PRELIMINARIES

2.1. Gauss genus theory. In this subsection, we will recall Gauss genus theory, which can be found in [Wan16, §3] for details. For our purpose, assume that $n = p_1 \cdots p_k \equiv 1 \pmod{4}$. Denote by $\mathcal{A}$ the ideal class group of $K = \mathbb{Q}(\sqrt{-n})$. Denote by $\mathcal{D}(K)$ the set of positive square-free divisors of $2n$. The classical Gauss genus theory tells that

$$\mathcal{A}[2] = \{(d, \sqrt{-n}) : d \in \mathcal{D}(K)\} \quad \text{and} \quad h_2(n) = \dim_{\mathbb{F}_2} \mathcal{A}[2] = k.$$

Denote by $p_{k+1} = 2$ and define the Rédei matrix

$$\mathbf{R}_n = ([p_j, -n]_{p_i})_{i,j} \in M_{k \times (k+1)}(\mathbb{F}_2). \quad (2.1)$$

Proposition 2.1 ([Rei34]). *We have*

$$\begin{array}{ccc} \text{Ker } \mathbf{R}_n & \xleftarrow{\sim} & \mathcal{D}(K) \cap \mathbf{N}_{K/\mathbb{Q}} K^\times & \longrightarrow & \mathcal{A}[2] \cap \mathcal{A}^2 \\ (v_{p_1}(d), \dots, v_{p_{k+1}}(d)) & \longleftarrow & d & \longmapsto & [(d, \sqrt{-n})], \end{array}$$

where the second arrow is a two-to-one onto homomorphism with kernel $\{1, n\}$. Hence $h_4(n) = k - \text{rank } \mathbf{R}_n$.

Proposition 2.2 ([Wan16, Proposition 3.6]). *For any $2^r d \in \mathcal{D}(K) \cap \mathbf{N}_{K/\mathbb{Q}} K^\times$ with odd d , let (α, β, γ) be a primitive triple of positive integers satisfying*

$$d\alpha^2 + \frac{n}{d}\beta^2 = 2^r\gamma^2.$$

Then $[(2^r d, \sqrt{-n})] \in \mathcal{A}^4$ if and only if

$$\mathbf{b}_\gamma = \left(\left[\frac{\gamma}{p_1} \right], \dots, \left[\frac{\gamma}{p_k} \right] \right)^T \in \text{Im } \mathbf{R}_n.$$

2.2. Torsion subgroup.

Proposition 2.3. *For any positive square-free integer n , $E_{\text{tor}}^{(n)}(\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$.*

We need the following lemma.

Lemma 2.4 ([Ono96]). *Let $\mathcal{E} : y^2 = x(x-a)(x+b)$ be an elliptic curve with $a, b \in \mathbb{Z}$.*

- (1) *$\mathcal{E}(\mathbb{Q})$ has a point of order 4 if and only if one of the three pairs $(-a, b)$, $(a, a+b)$ and $(-b, -a-b)$ consists of squares of integers.*

- (2) $\mathcal{E}(\mathbb{Q})$ has a point of order 3 if and only if there exist integers d, u, v such that $\gcd(u, v) = 1$, $d^2 u^3(u + 2v) = -a$, $d^2 v^3(v + 2u) = b$ and $u/v \notin \{-2, -1/2, -1, 1, 0\}$.

Proof of Proposition 2.3. Since $E^{(n)}$ has full rational 2-torsion, $E_{\text{tor}}^{(n)}(\mathbb{Q})$ contains a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. By Mazur's classification theorem [Maz77, Maz78], one has

$$E_{\text{tor}}^{(n)}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z}$$

for some $N \in \{1, 2, 3, 4\}$. We only need to show that $E^{(n)}(\mathbb{Q})$ contains no point of order 4 or 3. Since the three pairs in Lemma 2.4 (1) are $(-a^2 n, b^2 n)$, $(a^2 n, 2c^2 n)$ and $(-b^2 n, -2c^2 n)$, $E^{(n)}(\mathbb{Q})$ contains no point of order 4.

Assume that there are integers d, u, v such that $\gcd(u, v) = 1$,

$$d^2 u^3(u + 2v) = -a^2 n \quad \text{and} \quad d^2 v^3(v + 2u) = b^2 n.$$

Clearly, $d^2 = 1$ and $n = \gcd(u + 2v, v + 2u) = \gcd(3, u - v) = 1$ or 3 . Since a and b are odd, so is u, v . We may assume that $v > 0$, then $u < 0$. Since $n \mid (u + 2v, v + 2u)$, we may write $v = \alpha^2, u = -\beta^2$. Then $(\alpha^2 - 2\beta^2)/n$ and $(2\alpha^2 - \beta^2)/n$ are squares, which is impossible by modulo 8. Hence $E^{(n)}(\mathbb{Q})$ contains no point of order 3 by Lemma 2.4 (2). $\square$

2.3. Cassels pairing. As shown in [Cas98], the 2-Selmer group $\text{Sel}_2(E^{(n)})$ can be identified with

$$\{\Lambda = (d_1, d_2, d_3) \in (\mathbb{Q}^\times / \mathbb{Q}^{\times 2})^3 : D_\Lambda(\mathbb{A}_\mathbb{Q}) \neq \emptyset, d_1 d_2 d_3 \equiv 1 \pmod{\mathbb{Q}^{\times 2}}\},$$

where D_Λ is a genus one curve defined by

$$\begin{cases} H_1 : & -b^2 n t^2 + d_2 u_2^2 - d_3 u_3^2 = 0, \\ H_2 : & -a^2 n t^2 + d_3 u_3^2 - d_1 u_1^2 = 0, \\ H_3 : & 2c^2 n t^2 + d_1 u_1^2 - d_2 u_2^2 = 0. \end{cases} \quad (2.2)$$

Under this identification, the points $O, (a^2 n, 0), (-b^2 n, 0), (0, 0)$ and non-torsion $(x, y) \in E^{(n)}(\mathbb{Q})$ correspond to

$$(1, 1, 1), (2, 2n, n), (-2n, 2, -n), (-n, n, -1) \quad (2.3)$$

and $(x - a^2 n, x + b^2 n, x)$ respectively.

Cassels in [Cas98] defined an alternating bilinear pairing $\langle -, - \rangle$ on the $\mathbb{F}_2$ -vector space

$$\text{Sel}'_2(E^{(n)}) := \text{Sel}_2(E^{(n)}) / E^{(n)}(\mathbb{Q})[2]. \quad (2.4)$$

We will write it additively. For any $\Lambda \in \text{Sel}_2(E^{(n)})$, choose $P = (P_v) \in D_\Lambda(\mathbb{A}_\mathbb{Q})$. Since H_i is locally solvable everywhere, there exists $Q_i \in H_i(\mathbb{Q})$ by Hasse-Minkowski principle. Let L_i be a linear form in three variables such that $L_i = 0$ defines the tangent plane of H_i at Q_i . Then for any $\Lambda' = (d'_1, d'_2, d'_3) \in \text{Sel}_2(E^{(n)})$, define

$$\langle \Lambda, \Lambda' \rangle = \sum_v \langle \Lambda, \Lambda' \rangle_v \in \mathbb{F}_2, \quad \text{where} \quad \langle \Lambda, \Lambda' \rangle_v = \sum_{i=1}^3 [L_i(P_v), d'_i]_v.$$

This pairing is independent of the choice of P and Q_i , and is trivial on $E^{(n)}(\mathbb{Q})[2]$.

Lemma 2.5 ([Cas98, Lemma 7.2]). *Let p be an odd prime. Fix a choice of auxiliary points $Q_i \in H_i(\mathbb{Q})$ ($i = 1, 2, 3$) and of local representatives P_v . If the following two conditions hold simultaneously:*

- the coefficients of H_i and L_i are all integral at p for $i = 1, 2, 3$, and
- when D_Λ and L_i are reduced modulo p , they define a curve of genus 1 over $\mathbb{F}_p$ together with its tangent,

then the local Cassels pairing $\langle \Lambda, \Lambda' \rangle_p = 0$.

Lemma 2.6. *The following are equivalent:*

- (1) $\text{rank}_{\mathbb{Z}} E^{(n)}(\mathbb{Q}) = 0$ and $\text{III}(E^{(n)}/\mathbb{Q})[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^{2t}$;
- (2) $\text{Sel}'_2(E^{(n)}) \cong (\mathbb{Z}/2\mathbb{Z})^{2t}$ and the Cassels pairing on $\text{Sel}'_2(E^{(n)})$ is non-degenerate.

Proof. Note that $E^{(n)}(\mathbb{Q})[2] = (\mathbb{Z}/2\mathbb{Z})^2$ by [Proposition 2.3](#). The proof is similar to [\[Wan16, p. 2157\]](#). $\square$

By this lemma, the proof of our main result can be reduced to the calculations of the 2-Selmer group and the Cassels pairing on it.

3. 2-DESCENT METHOD

In this section, we will study the local solvability of homogeneous spaces and then express the 2-Selmer group as the null space of a matrix defined over $\mathbb{F}_2$.

3.1. Homogeneous spaces.

Lemma 3.1. *Let n be a positive square-free integer prime to $2abc$ and $\Lambda = (d_1, d_2, d_3)$, where d_1, d_2, d_3 are square-free integers and $d_1 d_2 d_3$ is a square.*

- (1) *If $p \nmid 2abcn$, then $D_\Lambda(\mathbb{Q}_p) \neq \emptyset$ if and only if $p \nmid d_1 d_2 d_3$.*
- (2) *If $D_\Lambda(\mathbb{Q}_2) \neq \emptyset$, then d_1 and d_2 have the same parity.*
- (3) *If both of d_1 and d_2 are odd, then $D_\Lambda(\mathbb{Q}_2) \neq \emptyset$ if and only if either $4 \mid d_1 - 1$, $8 \mid d_1 - d_2$ or $4 \mid d_1 + n$, $8 \mid d_1 - d_2 + 2n$.*
- (4) *$D_\Lambda(\mathbb{R}) \neq \emptyset$ if and only if $d_2 > 0$.*

Proof. Certainly, $\gcd(d_1, d_2, d_3) = 1$. Since we are dealing with homogeneous equations, we may assume that u_1, u_2, u_3 and t are p -adic integers and at least one of them is a p -adic unit.

(1) This follows from the classical descent theory (see [\[Sil09, Theorem X.1.1, Corollary X.4.4\]](#)).

(2) Suppose that $D_\Lambda(\mathbb{Q}_2) \neq \emptyset$. If $2 \mid d_1$, $2 \nmid d_2$, then $2 \mid d_3$. We have $2 \mid u_2$ by H_3 and $2 \mid t$ by H_1 . Then $2 \mid u_3$ by H_1 and $2 \mid u_1$ by H_2 , which is impossible. The case $2 \nmid d_1$, $2 \mid d_2$ is similar. Hence d_1 and d_2 have the same parity.

(3) If $D_\Lambda(\mathbb{Q}_2) \neq \emptyset$, then both of u_1, u_2 are odd by H_3 and exactly one of t and u_3 is even by H_2 . If t is even and u_3 is odd, then $4 \mid d_1 - d_3$, $8 \mid d_1 - d_2$ by $H_2 \bmod 4$ and $H_3 \bmod 8$. Note that if $8 \mid d_1 - d_2$, then $d_3 \equiv d_1 d_2 \equiv 1 \bmod 8$. If t is odd and u_3 is even, then $4 \mid d_1 + n$, $8 \mid d_1 - d_2 + 2n$ by $H_2 \bmod 4$ and $H_3 \bmod 8$.

Conversely, if $4 \mid d_1 - 1$, $8 \mid d_1 - d_2$, then $d_3 \equiv d_1 d_2 \equiv 1 \bmod 8$. Take

- $t = 0$, $u_1 = \sqrt{1/d_1}$, $u_2 = \sqrt{1/d_2}$, $u_3 = \sqrt{1/d_3}$ if $8 \mid d_1 - 1$;
- $t = 2$, $u_1 = 1$, $u_2 = \sqrt{(d_1 + 8c^2n)/d_2}$, $u_3 = \sqrt{(d_1 + 4a^2n)/d_3}$ if $8 \mid d_1 - 5$.

If $4 \mid d_1 + n$, $8 \mid d_1 - d_2 + 2n$, take

- $t = 1$, $u_1 = \sqrt{-a^2n/d_1}$, $u_2 = \sqrt{b^2n/d_2}$, $u_3 = 0$ if $8 \mid d_1 + n$;
- $t = 1$, $u_1 = \sqrt{(4d_3 - a^2n)/d_1}$, $u_2 = \sqrt{(4d_3 + b^2n)/d_2}$, $u_3 = 2$ if $8 \mid d_1 + n + 4$.

(4) Suppose that $D_\Lambda(\mathbb{R}) \neq \emptyset$. If $d_2 < 0$, then $d_3 < 0$ by H_1 . Thus $d_1 > 0$ by $d_1 d_2 d_3 \in \mathbb{Q}^{\times 2}$ but $d_1 < 0$ by H_2 , which is impossible. Hence $d_2 > 0$. The other direction is trivial. $\square$

Assume that n is a positive square-free integer prime to $2abc$. By [Lemma 3.1](#) and [Eq. \(2.3\)](#), any element of the pure 2-Selmer group $\text{Sel}'_2(E^{(n)})$ has a unique representative $\Lambda = (d_1, d_2, d_3)$, where d_1, d_2, d_3 are positive square-free integers dividing $nabc$. In the rest of this article, Λ is always assumed to be in this form and we will write $\Lambda = (d_1, d_2, d_3) \in \text{Sel}'_2(E^{(n)})$ for simplicity.

Lemma 3.2. *Let n be a positive square-free integer prime to $2abc$ and $\Lambda = (d_1, d_2, d_3)$. Let p be a prime factor of n . Then $D_\Lambda(\mathbb{Q}_p) \neq \emptyset$ if and only if*

- $\left(\frac{d_1}{p}\right) = \left(\frac{d_2}{p}\right) = 1$, if $p \nmid d_1, p \nmid d_2$;
- $\left(\frac{2d_1}{p}\right) = \left(\frac{2n/d_2}{p}\right) = 1$, if $p \nmid d_1, p \mid d_2$;
- $\left(\frac{-2n/d_1}{p}\right) = \left(\frac{2d_2}{p}\right) = 1$, if $p \mid d_1, p \nmid d_2$;
- $\left(\frac{-n/d_1}{p}\right) = \left(\frac{n/d_2}{p}\right) = 1$, if $p \mid d_1, p \mid d_2$.

Proof. Assume that $p \nmid d_1 d_2$, then $p \nmid d_3$. If $D_\Lambda(\mathbb{Q}_p) \neq \emptyset$, then $\left(\frac{d_2 d_3}{p}\right) = \left(\frac{d_1 d_3}{p}\right) = 1$ by H_2 and H_3 . That's to say, $\left(\frac{d_1}{p}\right) = \left(\frac{d_2}{p}\right) = 1$. Conversely, if $\left(\frac{d_1}{p}\right) = \left(\frac{d_2}{p}\right) = 1$, then $(0, \sqrt{1/d_1}, \sqrt{1/d_2}, \sqrt{1/d_3}) \in D_\Lambda(\mathbb{Q}_p)$. The rest of the cases can be proved similarly as in the congruent elliptic curve case, see [\[HB94, Appendix\]](#). $\square$

Lemma 3.3. *Let n be a positive square-free integer prime to $2abc$ and $\Lambda = (d_1, d_2, d_3)$. Let p be a prime factor of abc .*

- (1) *If $p \mid a$, then $D_\Lambda(\mathbb{Q}_p) \neq \emptyset$ if and only if one of the following cases holds:*
 - $p \nmid d_2, p \nmid d_1, \left(\frac{n}{p}\right) = -1$ or $\left(\frac{d_2}{p}\right) = 1$;
 - $p \nmid d_2, p \mid d_1, \left(\frac{d_2}{p}\right) = \left(\frac{n}{p}\right) = 1$.
- (2) *If $p \mid b$, then $D_\Lambda(\mathbb{Q}_p) \neq \emptyset$ if and only if one of the following cases holds:*
 - $p \nmid d_1, p \nmid d_2, \left(\frac{-n}{p}\right) = -1$ or $\left(\frac{d_1}{p}\right) = 1$;
 - $p \nmid d_1, p \mid d_2, \left(\frac{d_1}{p}\right) = \left(\frac{-n}{p}\right) = 1$.
- (3) *If $p \mid c$, then $D_\Lambda(\mathbb{Q}_p) \neq \emptyset$ if and only if one of the following cases holds:*
 - $p \nmid d_3, p \nmid d_1, \left(\frac{n}{p}\right) = -1$ or $\left(\frac{d_3}{p}\right) = 1$;
 - $p \nmid d_3, p \mid d_1, \left(\frac{d_3}{p}\right) = \left(\frac{n}{p}\right) = 1$.

Proof. Let p be a prime factor of a .

Suppose that $D_\Lambda(\mathbb{Q}_p) \neq \emptyset$. If $p \mid d_2$, then p divides exactly one of d_1 and d_3 . We may assume that $p \mid d_1$ and $p \nmid d_3$. Then p divides u_3, t by the solvability of H_2, H_3 and then u_2, u_1 by the solvability of H_1, H_2 . So $p \mid \gcd(t, u_1, u_2, u_3)$, which will cause a contradiction. Hence $p \nmid d_2$.

Suppose that $p \nmid d_1, p \nmid d_3$. If $D_\Lambda(\mathbb{Q}_p) \neq \emptyset$, then we may assume that $p \nmid \gcd(t, u_1, u_2, u_3)$. If $p \mid u_1$, then $p \mid u_3$, which implies that $\left(\frac{nd_2}{p}\right) = 1$ since

$H_1(\mathbb{Q}_p) \neq \emptyset$. If u_1 is a unit, then so is u_3 , which implies that $\left(\frac{d_1 d_3}{p}\right) = \left(\frac{d_2}{p}\right) = 1$ since $H_2(\mathbb{Q}_p) \neq \emptyset$.

Conversely, H_1 is a smooth conic curve over $\mathbb{F}_p$, which is guaranteed to have a point. We can lift it to $P = (t, u_2, u_3) \in H_1(\mathbb{Q}_p)$ by Hensel's lemma. Similarly,

$$H'_2 : -(ap^{-v_p(a)})^2 nt^2 + d_3 X^2 - d_1 Y^2 = 0$$

admits a point over $\mathbb{Q}_p$, which gives a point $Q = (t, u_3, u_1) \in H_2(\mathbb{Q}_p)$. If $\left(\frac{d_2}{p}\right) = 1$, then from values of t and u_3 taken from the point $P = (t, u_2, u_3) \in H_1(\mathbb{Q}_p)$, the equation H_2 has a solution over $\mathbb{Q}_p$; together with the point P , this yields a point in $D_\Lambda(\mathbb{Q}_p)$. Similarly, if $\left(\frac{nd_2}{p}\right) = 1$, then from values of t and u_3 taken from the point $Q = (t, u_3, u_1) \in H_2(\mathbb{Q}_p)$, the equation H_1 has a solution over $\mathbb{Q}_p$; together with the point Q , this yields a point in $D_\Lambda(\mathbb{Q}_p)$.

Suppose that $p \mid d_1, p \mid d_3$. If $D_\Lambda(\mathbb{Q}_p) \neq \emptyset$, then $\left(\frac{d_2 n}{p}\right) = 1$ by H_1 and $\left(\frac{d_2}{p}\right) = 1$ by H_2 . Conversely, if $\left(\frac{d_2}{p}\right) = \left(\frac{n}{p}\right) = 1$, then we may take $t = 1$ and

$$\begin{aligned} u_1 &= d_2 / \gcd(d_1, d_2), \\ u_3^2 &= d_2 + a^2 n / d_3 \equiv d_2 \pmod{p}, \\ u_2^2 &= d_3 + 2c^2 n / d_2 \equiv b^2 n / d_2 \pmod{p}. \end{aligned}$$

Hence $D_\Lambda(\mathbb{Q}_p)$ is non-empty.

The rest cases can be proved similarly. $\square$

Lemma 3.4. *Let n be a positive square-free integer prime to $2abc$ and $\Lambda = (d_1, d_2, d_3)$. If $D_\Lambda(\mathbb{Q}_v) \neq \emptyset$ for all places $v \neq 2$, then $D_\Lambda(\mathbb{Q}_2)$ is also non-empty.*

Proof. Since $D_\Lambda(\mathbb{Q}_v) \neq \emptyset$ for all places $v \neq 2$, each H_i is locally solvable at $v \neq 2$. By the product formula of Hilbert symbols, H_i is locally solvable at 2. In other words,

$$[nd_2, d_2 d_3]_2 = [-nd_1, d_3 d_1]_2 = [2nd_2, d_1 d_2]_2 = 0.$$

Then $[nd_2, d_1]_2 = [-nd_1, d_2]_2 = 0$.

- If $d_1 \equiv d_2 \pmod{4}$, then $[-n, d_1]_2 = [n, d_2]_2 = [2, d_1 d_2]_2 = 0$, which forces $4 \mid d_1 - 1$ and $8 \mid d_1 - d_2$.
- If $d_1 \equiv -d_2 \pmod{4}$, then $[n, d_1]_2 = [-n, -d_1]_2 = 0$ and $n \equiv -d_1 \equiv d_2 \pmod{4}$. Since $[2, d_1 d_2]_2 = [2nd_2, d_1 d_2]_2 = 0$, we have $d_1 d_2 \equiv -1 \pmod{8}$. In other words, $4 \mid d_1 + n$ and $8 \mid d_1 - d_2 + 2n$.

Hence $D_\Lambda(\mathbb{Q}_2) \neq \emptyset$ by [Lemma 3.1 \(3\)](#). $\square$

3.2. Matrix representation. By the results in the previous subsection, we can express the pure 2-Selmer group $\text{Sel}'_2(E^{(n)})$ as the kernel of a matrix. For our purpose, we assume that n is prime to abc and each prime factor of n is a quadratic residue modulo every prime factor of abc .

Denote by $n = p_1 \cdots p_k$ and

$$a = q_1^{t_1} \cdots q_{\ell_1}^{t_{\ell_1}}, \quad b = q_{\ell_1+1}^{t_{\ell_1+1}} \cdots q_{\ell_2}^{t_{\ell_2}}, \quad c = q_{\ell_2+1}^{t_{\ell_2+1}} \cdots q_{\ell}^{t_{\ell}}$$

the prime decompositions respectively, where all $t_i > 0$ and $0 \leq \ell_1 \leq \ell_2 \leq \ell$. We order the primes $q_{\ell_1+1}, \dots, q_{\ell_2}$ so that

$$q_{\ell_1+1}, \dots, q_{\ell_1+s} \equiv -1 \pmod{4}, \quad q_{\ell_1+s+1}, \dots, q_{\ell_2} \equiv 1 \pmod{4}.$$

Let $\Lambda = (d_1, d_2, d_3) \in \text{Sel}'_2(E^{(n)})$ where d_1, d_2, d_3 are positive square-free integers dividing $nabc$. By [Lemma 3.3](#), we have $\gcd(a, d_2) = \gcd(b, d_1) = \gcd(c, d_3) = 1$. In other words, $d_1 \mid nac$, $d_2 \mid nbc$ and $d_3 \mid nab$. So we may write

$$\begin{aligned} d_1 &= p_1^{x_1} \cdots p_k^{x_k} \cdot q_1^{z_1} \cdots q_{\ell_1}^{z_{\ell_1}} \cdot q_{\ell_2+1}^{z_{\ell_2+1}} \cdots q_{\ell}^{z_{\ell}}, \\ d_2 &= p_1^{y_1} \cdots p_k^{y_k} \cdot q_{\ell_1+1}^{z_{\ell_1+1}} \cdots q_{\ell_2}^{z_{\ell_2}} \cdot q_{\ell_2+1}^{z_{\ell_2+1}} \cdots q_{\ell}^{z_{\ell}}, \\ d_3 &\equiv p_1^{x_1+y_1} \cdots p_k^{x_k+y_k} \cdot q_1^{z_1} \cdots q_{\ell_1}^{z_{\ell_1}} \cdot q_{\ell_1+1}^{z_{\ell_1+1}} \cdots q_{\ell_2}^{z_{\ell_2}} \pmod{\mathbb{Q}^{\times 2}}. \end{aligned}$$

Denote by

$$\mathbf{x} = (x_1, \dots, x_k)^T, \quad \mathbf{y} = (y_1, \dots, y_k)^T \in \mathbb{F}_2^k,$$

and

$$\mathbf{z} = (z_1, \dots, z_{\ell_1}, z_{\ell_1+1}, \dots, z_{\ell_2}, z_{\ell_2+1}, \dots, z_{\ell})^T \in \mathbb{F}_2^{\ell}.$$

Denote by

$$\begin{pmatrix} \mathbf{F}_1 & \mathbf{F}_2 & \mathbf{F}_3 \\ \mathbf{F}_4 & \mathbf{F}_5 & \mathbf{F}_6 \\ \mathbf{F}_7 & \mathbf{F}_8 & \mathbf{F}_9 \end{pmatrix} = ([q_j, q_i]_{q_i})_{i,j} \in M_{\ell}(\mathbb{F}_2),$$

where $\mathbf{F}_1 \in M_{\ell_1}(\mathbb{F}_2)$ and $\mathbf{F}_5 \in M_{\ell_2-\ell_1}(\mathbb{F}_2)$. Denote by

$$\mathcal{M}_1 = \begin{pmatrix} & \mathbf{F}_2 & \mathbf{F}_3 \\ \mathbf{O} & \mathbf{I}_s & \mathbf{O} & \mathbf{O} \\ \mathbf{F}'_4 & \mathbf{O} & \mathbf{O} & \mathbf{F}'_6 \\ \mathbf{F}_7 & \mathbf{F}_8 & & \end{pmatrix} \in M_{\ell}(\mathbb{F}_2),$$

where $\mathbf{F}'_4$ (resp. $\mathbf{F}'_6$) is the $(k+1)$ to $(\ell_2 - \ell_1)$ rows of $\mathbf{F}_4$ (resp. $\mathbf{F}_6$).

Lemma 3.5. *Notations as above. The map $(d_1, d_2, d_3) \mapsto \mathbf{z}$ induces an isomorphism*

$$\text{Sel}'_2(E) \xrightarrow{\sim} \text{Ker } \mathcal{M}_1.$$

Proof. In the language of linear algebra, [Lemma 3.3](#) tells that

- (1) $(\mathbf{O}, \mathbf{F}_2, \mathbf{F}_3)\mathbf{z} = \mathbf{0}$;
- (2) if $\ell_1 + 1 \leq i \leq \ell_1 + s$, then $q_i \nmid d_2$, $z_i = 0$,

$$(\mathbf{O}_{s \times \ell_1}, \mathbf{I}_s, \mathbf{O}_{s \times (\ell_2 - \ell_1 - s)})\mathbf{z} = \mathbf{0};$$

$$\text{if } \ell_1 + s \leq i \leq \ell_2, \text{ then } \left(\frac{-1}{q_i}\right) = 1, \left(\frac{d_1}{p}\right) = 1,$$

$$(\mathbf{F}'_4, \mathbf{O}, \mathbf{F}'_6)\mathbf{z} = \mathbf{0};$$

- (3) $(\mathbf{F}_7, \mathbf{F}_8, \mathbf{O})\mathbf{z} = \mathbf{0}$.

The result then follows from [Lemma 3.1 \(4\)](#) and [Lemma 3.4](#) by noting that $n = 1$. $\square$

Denote by

$$\begin{aligned} \mathbf{D}_u &= \text{diag} \left\{ \left[\frac{u}{p_1} \right], \dots, \left[\frac{u}{p_k} \right] \right\} \in M_k(\mathbb{F}_2), \\ \mathbf{A} = \mathbf{A}_n &= ([p_j, -n]_{p_i})_{i,j} \in M_k(\mathbb{F}_2) \end{aligned} \tag{3.1}$$

and

$$(\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3) = ([q_j, -n]_{p_i})_{i,j} \in M_{k \times \ell}(\mathbb{F}_2),$$

where $\mathbf{G}_1 \in M_{k \times \ell_1}(\mathbb{F}_2)$ and $\mathbf{G}_2 \in M_{k \times (\ell_2 - \ell_1)}(\mathbb{F}_2)$. Denote the Monsky matrix by

$$\mathbf{M}_n = \begin{pmatrix} \mathbf{A} + \mathbf{D}_{-2} & \mathbf{D}_2 \\ \mathbf{D}_2 & \mathbf{A} + \mathbf{D}_2 \end{pmatrix} \quad (3.2)$$

and the generalized Monsky matrix by

$$\mathcal{M}_n = \begin{pmatrix} \mathbf{M}_n & \mathbf{G} \\ \mathcal{M}_1 & \end{pmatrix}, \quad \text{where } \mathbf{G} = \begin{pmatrix} \mathbf{G}_1 & \mathbf{G}_3 \\ \mathbf{G}_2 & \mathbf{G}_3 \end{pmatrix}. \quad (3.3)$$

See [HB94, Appendix].

Proposition 3.6. *Notations as above. The map $(d_1, d_2, d_3) \mapsto \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{pmatrix}$ induces an isomorphism*

$$\text{Sel}'_2(E^{(n)}) \xrightarrow{\sim} \text{Ker } \mathcal{M}_n.$$

Proof. This follows from Lemma 3.1 (4), Lemma 3.2, Lemma 3.3, Lemma 3.4, and Lemma 3.5 with $\left(\frac{n}{q}\right) = 1$. $\square$

4. SECOND MINIMAL SHAFAREVICH-TATE GROUP

In this section, we will prove Theorem 1.1 by calculating the Cassels pairing. Let $n = p_1 \cdots p_k \equiv 1 \pmod{8}$ be a positive square-free integer prime to abc where each p_i is a quadratic residue modulo every prime factor of abc .

4.1. Proof of Theorem 1.1 (A).

Lemma 4.1. *Assume that each $p_i \equiv \pm 1 \pmod{8}$. Let $\mathbf{d} = (s_1, \dots, s_k)^T$ be a column vector in $\mathbb{F}_2^k$ and $d = p_1^{s_1} \cdots p_k^{s_k}$.*

- (1) $\mathbf{d} \in \text{Ker}(\mathbf{A} + \mathbf{D}_{-1})$ if and only if $\mathbf{d} + \left[\frac{-1}{d}\right] \mathbf{1} \in \text{Ker } \mathbf{A}^T$.
- (2) Assume that $\text{Sel}_2(E/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$. Then $\dim_{\mathbb{F}_2} \text{Sel}'_2(E^{(n)}) = 2$ if and only if $h_4(n) = 1$. In which case, $\text{Sel}'_2(E^{(n)})$ is generated by $(2, 2, 1)$ and $(d, 1, d)$, where $\text{Ker}(\mathbf{A} + \mathbf{D}_{-1}) = \{\mathbf{0}, \mathbf{d}\}$.

Proof. (1) We may rearrange the ordering of the prime factors p_i such that $p_1 \equiv \cdots \equiv p_{k'} \equiv -1 \pmod{8}$ and $p_{k'+1} \equiv \cdots \equiv p_k \equiv 1 \pmod{8}$. Then $\mathbf{b}_{-1} = \begin{pmatrix} \mathbf{1}' \\ \mathbf{0} \end{pmatrix}$, where $\mathbf{1}' \in \mathbb{F}_2^{k'}$. By the quadratic reciprocity law, one can show that

$$\mathbf{A}^T = \mathbf{A} + \mathbf{D}_{-1} + \mathbf{b}_{-1} \mathbf{b}_{-1}^T.$$

Since $n \equiv 1 \pmod{8}$, k' is even and $\mathbf{b}_{-1}^T \mathbf{1} = \mathbf{1}^T \mathbf{b}_{-1} = \mathbf{b}_{-1}^T \mathbf{b}_{-1} = k' = 0 \in \mathbb{F}_2$. Since $\mathbf{A} \mathbf{1} = \mathbf{0}$, we have

$$\mathbf{A}^T \mathbf{1} = (\mathbf{A} + \mathbf{D}_{-1} + \mathbf{b}_{-1} \mathbf{b}_{-1}^T) \mathbf{1} = \mathbf{b}_{-1}$$

and

$$\mathbf{A}^T (\mathbf{I} + \mathbf{1} \mathbf{b}_{-1}^T) = \mathbf{A}^T + \mathbf{b}_{-1} \mathbf{b}_{-1}^T = \mathbf{A} + \mathbf{D}_{-1}.$$

Hence $\mathbf{d} \in \text{Ker}(\mathbf{A} + \mathbf{D}_{-1})$ if and only if

$$(\mathbf{I} + \mathbf{1} \mathbf{b}_{-1}^T) \mathbf{d} = \mathbf{d} + (\mathbf{b}_{-1}^T \mathbf{d}) \mathbf{1} = \mathbf{d} + \left[\frac{-1}{d}\right] \mathbf{1} \in \text{Ker } \mathbf{A}^T.$$

(2) Since $\dim_{\mathbb{F}_2} \text{Sel}'_2(E) = 0$, we have $\text{Ker } \mathcal{M}_1 = 0$ by [Lemma 3.5](#). By [Proposition 3.6](#), $\dim_{\mathbb{F}_2} \text{Sel}'_2(E^{(n)}) = 2$ if and only if the rank of

$$\mathbf{M}_n = \text{diag}\{\mathbf{A} + \mathbf{D}_{-1}, \mathbf{A}\}$$

is $2k - 2$. By [\(1\)](#), we have $\text{rank } \mathbf{A} = \text{rank}(\mathbf{A} + \mathbf{D}_{-1})$ and then

$$\dim_{\mathbb{F}_2} \text{Sel}'_2(E^{(n)}) = 2 \iff \text{rank } \mathbf{A} = k - 1.$$

Note that the Rédei matrix of $\mathbb{Q}(\sqrt{-n})$ is $\mathbf{R}_n = (\mathbf{A}, \mathbf{0})$. Then $h_4(n) = 1$ if and only if $\text{rank } \mathbf{A} = k - 1$ by [Proposition 2.1](#).

If $\text{rank } \mathbf{A} = k - 1$, then $\text{Ker } \mathbf{A} = \{\mathbf{0}, \mathbf{1}\}$. Hence

$$\text{Ker } \mathcal{M}_n = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} d \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} d \\ 1 \\ 0 \end{pmatrix} \right\}.$$

In other words, $\text{Sel}'_2(E^{(n)})$ is generated by $(1, n, n)$ and $(d, 1, d)$. Conclude the result by the fact that $(1, n, n) - (2, 2, 1) = (2, 2n, n)$ corresponds to a torsion point, see [Eq. \(2.3\)](#). $\square$

Theorem 4.2. *Assume that $\text{Sel}_2(E/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$. Let n be a positive square-free integer prime to abc where each prime factor of n is a quadratic residue modulo every prime factor of abc . If $n \equiv 1 \pmod{8}$ and all of its prime factors are congruent to ± 1 modulo 8, then the following are equivalent:*

- (1) $\text{rank}_{\mathbb{Z}} E^{(n)}(\mathbb{Q}) = 0$ and $\text{III}(E^{(n)}/\mathbb{Q})[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2$;
- (2) $h_4(n) = 1$ and $h_8(n) = 0$.

Proof. By [Lemma 2.6](#), [\(1\)](#) is equivalent to say, $\text{Sel}'_2(E^{(n)})$ has dimension 2 and the Cassels pairing on it is non-degenerate. By [Lemma 4.1 \(2\)](#), $\dim_{\mathbb{F}_2} \text{Sel}'_2(E^{(n)}) = 2$ if and only if $h_4(n) = 1$.

Since all prime factors of n are congruent to ± 1 modulo 8, 2 is a norm from $\mathbb{Q}(\sqrt{-n})$ and there exists a primitive triple (α, β, γ) of positive integers such that

$$\alpha^2 + n\beta^2 = 2\gamma^2.$$

It's easy to see that all of α, β, γ are odd.

Assume that $h_4(n) = 1$. Then by [Lemma 4.1 \(2\)](#), $\text{Sel}'_2(E^{(n)})$ is generated by $\Lambda = (2, 2, 1)$ and $\Lambda' = (d, 1, d)$. Recall that D_Λ is

$$\begin{cases} H_1 : & -b^2nt^2 + 2u_2^2 - u_3^2 = 0, \\ H_2 : & -a^2nt^2 + u_3^2 - 2u_1^2 = 0, \\ H_3 : & c^2nt^2 + u_1^2 - u_2^2 = 0. \end{cases}$$

Choose

$$\begin{aligned} Q_1 &= (\beta, b\gamma, b\alpha) \in H_1(\mathbb{Q}), & L_1 &= bn\beta t - 2\gamma u_2 + \alpha u_3, \\ Q_3 &= (0, 1, 1) \in H_3(\mathbb{Q}), & L_3 &= u_1 - u_2. \end{aligned}$$

By [Lemma 2.5](#), we have

$$\langle \Lambda, \Lambda' \rangle = \sum_{p|2nabc} [L_1 L_3(P_p), d]_p$$

for any $P_p \in D_\Lambda(\mathbb{Q}_p)$. Since $\left(\frac{p_i}{q}\right) = 1$ for any prime $q \mid abc$, we have $\left(\frac{d}{q}\right) = 1$ and $\langle \Lambda, \Lambda' \rangle_q = 0$.

For $p \mid n$, $\alpha^2 \equiv 2\gamma^2 \pmod{p}$. We may take $\sqrt{2} \in \mathbb{Q}_p$ such that $\sqrt{2}\gamma \equiv \alpha \pmod{p}$. Take $P_p = (t, u_1, u_2, u_3) = (0, 1, -1, \sqrt{2})$, then

$$L_1 L_3(P_p) = 2(2\gamma + \sqrt{2}\alpha) \equiv 8\gamma \pmod{p}$$

and

$$\langle \Lambda, \Lambda' \rangle_p = [L_1 L_3(P_p), d]_p = [\gamma, d]_p.$$

Note that

$$n(b\beta)^2 - (a\alpha)^2 = 2(b^2\gamma^2 - c^2\alpha^2) \equiv 0 \pmod{16},$$

we may take $\sqrt{n} \in \mathbb{Q}_2$ such that $b\beta\sqrt{n} \equiv a\alpha \pmod{8}$. Take $P_2 = (1, 0, c\sqrt{n}, -a\sqrt{n})$, then

$$L_1 L_3(P_2) = -c\sqrt{n}(bn\beta - 2c\gamma\sqrt{n} - a\alpha\sqrt{n}) = 2c^2n\gamma + cn(a\alpha - b\beta\sqrt{n})$$

and

$$\langle \Lambda, \Lambda' \rangle_2 = [L_1 L_3(P_2), d]_2 = [2c^2n\gamma, d]_2 = [\gamma, d]_2 = \left[\frac{-1}{d} \right] \left[\frac{-1}{\gamma} \right].$$

Since $\alpha^2 \equiv -n\beta^2 \pmod{\gamma}$, we have $\left(\frac{-1}{\gamma} \right) = \left(\frac{n}{\gamma} \right) = \left(\frac{\gamma}{n} \right)$. Hence

$$\langle \Lambda, \Lambda' \rangle = \sum_{p \mid n} \langle \Lambda, \Lambda' \rangle_p + \langle \Lambda, \Lambda' \rangle_2 = \left[\frac{\gamma}{d} \right] + \left[\frac{-1}{d} \right] \left[\frac{\gamma}{n} \right].$$

Since $\mathbf{R}_n = (\mathbf{A}, \mathbf{0})$, we have $\mathcal{A}[2] \cap \mathcal{A}^2 = \{[(1)], [(2, \sqrt{-n})]\}$. Since

$$\text{Ker } \mathbf{A}^T = \left\{ \mathbf{0}, \mathbf{d} + \left[\frac{-1}{d} \right] \mathbf{1} \right\}$$

by [Lemma 4.1 \(1\)](#), we have

$$\text{Im } \mathbf{R}_n = \text{Im } \mathbf{A} = \left\{ \mathbf{u} : \mathbf{u}^T \left(\mathbf{d} + \left[\frac{-1}{d} \right] \mathbf{1} \right) = 0 \right\}.$$

By [Proposition 2.2](#), $[(2, \sqrt{-n})] \in \mathcal{A}^4$ if and only if

$$\mathbf{b}_\gamma = \left(\left[\frac{\gamma}{p_1} \right], \dots, \left[\frac{\gamma}{p_k} \right] \right)^T \in \text{Im } \mathbf{R}_n,$$

if and only if

$$\langle \Lambda, \Lambda' \rangle = \left[\frac{\gamma}{d} \right] + \left[\frac{-1}{d} \right] \left[\frac{\gamma}{n} \right] = \mathbf{b}_\gamma^T \left(\mathbf{d} + \left[\frac{-1}{d} \right] \mathbf{1} \right) = 0.$$

In conclusion, the Cassels pairing is non-degenerate if and only if $h_8(n) = 0$. $\square$

4.2. Proof of [Theorem 1.1 \(B\)](#).

Lemma 4.3. Assume that each $p_i \equiv 1 \pmod{4}$ and $\text{Sel}_2(E/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$. Let $\mathbf{d} = (s_1, \dots, s_k)^T$ be a column vector in $\mathbb{F}_2^k$ and $d = p_1^{s_1} \cdots p_k^{s_k}$.

- (1) $\dim_{\mathbb{F}_2} \text{Sel}'_2(E^{(n)}) = 2$ if and only if $h_4(n) = 1$. In which case, $\text{rank } \mathbf{A} = k-2$ or $k-1$.
- (2) If $h_4(n) = 1$ and $\text{rank } \mathbf{A} = k-2$, then $\text{Sel}'_2(E^{(n)})$ is generated by $(d, d, 1)$ and $(-1, 1, -1)$, where $\text{Ker } \mathbf{A} = \{\mathbf{0}, \mathbf{1}, \mathbf{d}, \mathbf{d} + \mathbf{1}\}$. Moreover, $d \equiv 5 \pmod{8}$.
- (3) If $h_4(n) = 1$ and $\text{rank } \mathbf{A} = k-1$, then $\text{Sel}'_2(E^{(n)})$ is generated by $(2d, 2d, 1)$ and $(-1, 1, -1)$, where $\mathbf{A}\mathbf{d} = \mathbf{b}_2$.

Proof. Similar to the proof of [Lemma 4.1 \(2\)](#), we have $\text{Ker } \mathcal{M}_1 = 0$. It suffices to show that $\text{rank } \mathbf{M}_n = 2k - 2$ if and only if the Rédei matrix $\mathbf{R}_n = (\mathbf{A}, \mathbf{b}_2)$ has rank $k - 1$ by [Proposition 2.1](#). Since $\mathbf{A}\mathbf{1} = \mathbf{0}$, we have $\text{rank } \mathbf{A} \leq k - 1$. If $\text{rank } \mathbf{M}_n = 2k - 2$, then

$$2k - 2 = \text{rank} \begin{pmatrix} \mathbf{A} + \mathbf{D}_2 & \mathbf{D}_2 \\ \mathbf{D}_2 & \mathbf{A} + \mathbf{D}_2 \end{pmatrix} = \text{rank} \begin{pmatrix} \mathbf{A} & \mathbf{D}_2 \\ \mathbf{D}_2 & \mathbf{A} \end{pmatrix} \leq k + \text{rank } \mathbf{A}$$

and $\text{rank } \mathbf{A} \geq k - 2$. If $\text{rank } \mathbf{R}_n = k - 1$, then clearly $\text{rank } \mathbf{A} \geq k - 2$.

Suppose that $\text{rank } \mathbf{A} = k - 2$. If $\text{rank } \mathbf{M}_n = 2k - 2$, then $\mathbf{b}_2 \notin \text{Im } \mathbf{A}$. Otherwise assume that $\mathbf{A}\mathbf{a} = \mathbf{b}_2$, then

$$\text{Ker } \mathbf{M}_n \supseteq \left\{ \begin{pmatrix} \mathbf{u} \\ \mathbf{u} \end{pmatrix}, \begin{pmatrix} \mathbf{u} + \mathbf{a} \\ \mathbf{u} + \mathbf{a} + \mathbf{1} \end{pmatrix} : \mathbf{u} \in \text{Ker } \mathbf{A} \right\}$$

has at least 8 elements, which is impossible. Therefore, $\text{rank } \mathbf{R}_n = \text{rank}(\mathbf{A}, \mathbf{b}_2) = k - 1$. Conversely, if $\text{rank } \mathbf{R}_n = k - 1$, then $\mathbf{b}_2 \notin \text{Im } \mathbf{A}$. Since $n \equiv 1 \pmod{8}$, we have $\mathbf{1}^T \mathbf{b}_2 = 0$. Note that $\mathbf{A}$ is symmetric, we have

$$\text{Im } \mathbf{A} = \{\mathbf{u} : \mathbf{1}^T \mathbf{u} = \mathbf{d}^T \mathbf{u} = 0\},$$

$\mathbf{d}^T \mathbf{b}_2 = 1$ and $\mathbf{1}^T \mathbf{D}_2(\mathbf{d} + \mathbf{1}) = \mathbf{1}^T \mathbf{D}_2 \mathbf{d} = \mathbf{b}_2^T \mathbf{d} = 1$. Hence $\mathbf{D}_2 \mathbf{1}, \mathbf{D}_2 \mathbf{d}, \mathbf{D}_2(\mathbf{d} + \mathbf{1}) \notin \text{Im } \mathbf{A}$. If $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \in \text{Ker } \mathbf{M}_n$, then $\mathbf{x} + \mathbf{y} \in \text{Ker } \mathbf{A}$ and $\mathbf{D}_2(\mathbf{x} + \mathbf{y}) = \mathbf{A}\mathbf{x}$. This forces $\mathbf{x} + \mathbf{y} = \mathbf{0}$ and $\mathbf{x} = \mathbf{y} \in \text{Ker } \mathbf{A}$. Hence $\#\text{Ker } \mathbf{M}_n = \#\text{Ker } \mathbf{A} = 4$ and $\text{rank } \mathbf{M}_n = 2k - 2$. In this case,

$$\text{Ker } \mathcal{M}_n = \left\{ \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{d} \\ \mathbf{d} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{d} + \mathbf{1} \\ \mathbf{d} + \mathbf{1} \\ \mathbf{0} \end{pmatrix} \right\}.$$

In other words, $\text{Sel}'_2(E^{(n)})$ is generated by $(n, n, 1)$ and $(d, d, 1)$. Since $\mathbf{d}^T \mathbf{b}_2 = 1$, we have $\begin{bmatrix} 2 \\ d \end{bmatrix} = 1$ and $d \equiv 5 \pmod{8}$.

Suppose that $\text{rank } \mathbf{A} = k - 1$. Then $\text{Ker } \mathbf{A} = \{\mathbf{0}, \mathbf{1}\}$ and $\text{Im } \mathbf{A} = \{\mathbf{u} : \mathbf{1}^T \mathbf{u} = 0\}$. Since $n \equiv 1 \pmod{8}$, we have $\mathbf{1}^T \mathbf{b}_2 = 0$ and $\mathbf{b}_2 \in \text{Im } \mathbf{A}$. Thus $\text{rank } \mathbf{R}_n = k - 1$, $h_4(n) = 1$ and

$$\text{Ker } \mathcal{M}_n = \left\{ \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{d} \\ \mathbf{d} + \mathbf{1} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{d} + \mathbf{1} \\ \mathbf{d} \\ \mathbf{0} \end{pmatrix} \right\}.$$

In this case, $\text{Sel}'_2(E^{(n)})$ is generated by $(n, n, 1)$ and (d, nd, n) .

Conclude the result by the fact that $(n, n, 1) - (-1, 1, -1) = (-n, n, -1)$ and $(d, nd, n) - (2d, 2d, 1) = (2, 2n, n)$ correspond torsions, see [Eq. \(2.3\)](#). $\square$

Theorem 4.4. Assume that $\text{Sel}_2(E/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$. Let n be a positive square-free integer prime to abc where each prime factor of n is a quadratic residue modulo every prime factor of abc . If $n \equiv 1 \pmod{8}$ and all of its prime factors are congruent to 1 modulo 4, then the following are equivalent:

- (1) $\text{rank}_{\mathbb{Z}} E^{(n)}(\mathbb{Q}) = 0$ and $\text{III}(E^{(n)}/\mathbb{Q})[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2$;
- (2) $h_4(n) = 1$ and $h_8(n) \equiv \frac{d-1}{4} \pmod{2}$.

Here d is the odd part of $d_0 \mid 2n$ such that the ideal class $[(d_0, \sqrt{-n})]$ is the non-trivial element in $\mathcal{A}[2] \cap \mathcal{A}^2$.

Proof. By Lemma 2.6, (1) is equivalent to say, $\text{Sel}'_2(E^{(n)})$ has dimension 2 and the Cassels pairing on it is non-degenerate. By Lemma 4.3 (1), $\dim_{\mathbb{F}_2} \text{Sel}'_2(E^{(n)}) = 2$ if and only if $h_4(n) = 1$. Assume that $h_4(n) = 1$.

(1) The case $\text{rank } \mathbf{A} = k - 2$. By Lemma 4.3 (2) and Proposition 2.1, we have $\mathbf{b}_2 \notin \text{Im } \mathbf{A}$ and $\mathcal{D}(K) \cap \mathbf{N}_{K/\mathbb{Q}} K^\times = \{1, n, d, n/d\}$ with $d = d_0 \equiv 5 \pmod{8}$. Denote by $d' = n/d \equiv 5 \pmod{8}$. Since d is a norm, there exists a primitive triple (α, β, γ) of positive integers such that

$$d\alpha^2 + d'\beta^2 = \gamma^2.$$

If α is odd, then β is even and the triple

$$(\alpha', \beta', \gamma') = \left(\left| \frac{(d-d')\alpha}{2} + d'\beta \right|, \left| \frac{(d-d')\beta}{2} - d\alpha \right|, \frac{(d+d')\gamma}{2} \right)$$

is another primitive solution with even α' . Thus we may assume that α is even. Then all of $\alpha/2, \beta, \gamma$ are odd since $d' \equiv 5 \pmod{8}$.

By Lemma 4.3 (2), $\text{Sel}'_2(E^{(n)})$ is generated by $\Lambda = (d, d, 1)$ and $\Lambda' = (-1, 1, -1)$. Recall that D_Λ is

$$\begin{cases} H_1 : & -b^2nt^2 + du_2^2 - u_3^2 = 0, \\ H_2 : & -a^2nt^2 + u_3^2 - du_1^2 = 0, \\ H_3 : & 2c^2d't^2 + u_1^2 - u_2^2 = 0. \end{cases}$$

Choose

$$\begin{aligned} Q_1 &= (\beta, b\gamma, bd\alpha) \in H_1(\mathbb{Q}), & L_1 &= bd'\beta t - \gamma u_2 + \alpha u_3, \\ Q_3 &= (0, 1, 1) \in H_3(\mathbb{Q}), & L_3 &= u_1 - u_2. \end{aligned}$$

By Lemma 2.5, we have

$$\langle \Lambda, \Lambda' \rangle = \sum_{p|2nabc\infty} [L_1 L_3(P_p), -1]_p$$

for any $P_p \in D_\Lambda(\mathbb{Q}_p)$. For each $p \mid n$, we have $p \equiv 1 \pmod{4}$ and then $\langle \Lambda, \Lambda' \rangle_p = 0$. Since for any $q \mid c$, we have $-a^2 = b^2 - 2c^2 \equiv b^2 \pmod{q}$, we have $q \equiv 1 \pmod{4}$ and then $\langle \Lambda, \Lambda' \rangle_q = 0$.

Take $P_\infty = (t, u_1, u_2, u_3) = (0, 1, -1, \sqrt{d})$, then

$$L_1 L_3(P_\infty) = 2(\gamma + \alpha\sqrt{d}) > 0$$

and

$$\langle \Lambda, \Lambda' \rangle_\infty = [L_1 L_3(P_\infty), -1]_\infty = 0.$$

Take $P_2 = (2, \sqrt{1-8c^2d'}, 1, \sqrt{d-4b^2n})$ where $u_1 \equiv 3 \pmod{8}$. Note that $bd'\beta + \alpha u_3/2$ is even. We have

$$L_1 L_3(P_2) = (u_1 - 1)(2bd'\beta + \alpha u_3 - \gamma)$$

and

$$\langle \Lambda, \Lambda' \rangle_2 = [L_1 L_3(P_2), -1]_2 = [2, -1]_2 + [-\gamma, -1]_2 = \left[\frac{-1}{\gamma} \right] + 1.$$

Since $d\alpha^2 \equiv -d'\beta^2 \pmod{\gamma}$, we have $\left(\frac{-1}{\gamma} \right) = \left(\frac{n}{\gamma} \right) = \left(\frac{\gamma}{n} \right)$ and

$$\langle \Lambda, \Lambda' \rangle_2 = \left[\frac{\gamma}{n} \right] + 1.$$

For $q \mid ab$, take $P_q = (0, 1, -1, \sqrt{d})$. Since $\gamma^2 - d\alpha^2 = d'\beta^2$, we may choose $\sqrt{d}$ such that $q \mid (\gamma - \alpha\sqrt{d})$ if $q \mid \beta$. Then

$$L_1 L_3(P_q) = 2(\gamma + \alpha\sqrt{d}) \in \mathbb{Z}_q^\times$$

and

$$\langle \Lambda, \Lambda' \rangle_q = [L_1 L_3(P_q), -1]_q = 0.$$

Hence

$$\langle \Lambda, \Lambda' \rangle = \langle \Lambda, \Lambda' \rangle_2 = \left[\frac{\gamma}{n} \right] + 1.$$

Since $\mathbf{R}_n = (\mathbf{A}, \mathbf{b}_2)$, we have $\mathcal{A}[2] \cap \mathcal{A}^2 = \{[(1)], [(d, \sqrt{-n})]\}$. Since $\mathbf{b}_2 \notin \text{Im } \mathbf{A}$ and $\mathbf{A}\mathbf{1} = \mathbf{0}$, we have

$$\text{Im } \mathbf{R}_n = \{\mathbf{u} : \mathbf{1}^T \mathbf{u} = 0\}.$$

By [Proposition 2.2](#), $[(d, \sqrt{-n})] \in \mathcal{A}^4$ if and only if

$$\mathbf{b}_\gamma = \left(\left[\frac{\gamma}{p_1} \right], \dots, \left[\frac{\gamma}{p_k} \right] \right)^T \in \text{Im } \mathbf{R}_n,$$

if and only if

$$\langle \Lambda, \Lambda' \rangle = \left[\frac{\gamma}{n} \right] + 1 = \mathbf{1}^T \mathbf{b}_\gamma + 1 = 1.$$

In conclusion, the Cassels pairing is non-degenerate if and only if $h_8(n) = 1 = \left[\frac{2}{d} \right]$.

(2) The case $\text{rank } \mathbf{A} = k - 1$. By [Lemma 4.3 \(3\)](#) and [Proposition 2.1](#), we have $\mathbf{b}_2 \in \text{Im } \mathbf{A}$ and $\mathcal{D}(K) \cap \mathbf{N}_{K/\mathbb{Q}} K^\times = \{1, n, 2d, 2n/d\}$. Denote by $d' = n/d$. Since $d_0 = 2d$ is a norm, there exists a primitive triple (α, β, γ) of positive integers such that

$$d\alpha^2 + d'\beta^2 = 2\gamma^2.$$

It's easy to see that all of α, β, γ are odd.

The group $\text{Sel}_2'(E^{(n)})$ is generated by $\Lambda = (2d, 2d, 1)$ and $\Lambda' = (-1, 1, -1)$ by [Lemma 4.3 \(3\)](#). Recall that D_Λ is

$$\begin{cases} H_1 : & -b^2 n t^2 + 2du_2^2 - u_3^2 = 0, \\ H_2 : & -a^2 n t^2 + u_3^2 - 2du_1^2 = 0, \\ H_3 : & c^2 d' t^2 + u_1^2 - u_2^2 = 0. \end{cases}$$

Choose

$$\begin{aligned} Q_1 &= (\beta, b\gamma, bd\alpha) \in H_1(\mathbb{Q}), & L_1 &= bd'\beta t - 2\gamma u_2 + \alpha u_3, \\ Q_3 &= (0, 1, 1) \in H_3(\mathbb{Q}), & L_3 &= u_1 - u_2. \end{aligned}$$

Similar to the case $\text{rank } \mathbf{A} = k - 2$, we have

$$\langle \Lambda, \Lambda' \rangle = \sum_{p \mid 2ab\infty} [L_1 L_3(P_p), -1]_p$$

for any $P_p \in D_\Lambda(\mathbb{Q}_p)$.

For $p = \infty$, take $P_\infty = (0, 1, -1, \sqrt{2d})$. Then

$$L_1 L_3(P_\infty) = 2(2\gamma + \alpha\sqrt{2d}) > 0$$

and

$$\langle \Lambda, \Lambda' \rangle_\infty = [L_1 L_3(P_\infty), -1]_\infty = 0.$$

For $p = 2$, take $P_2 = (t, u_1, u_2, u_3)$ where

$$t = 1, u_1 = 2\left[\frac{2}{d}\right], u_2^2 = c^2 d' + u_1^2, u_3^2 = a^2 n + 2du_1^2$$

with $\gamma u_2 \equiv 1 \pmod{4}$. Since

$$\begin{aligned} (bd'\beta + \alpha u_3)(bd'\beta - \alpha u_3) &= b^2 d'^2 \beta^2 - \alpha^2 (a^2 n + 2du_1^2) \\ &= b^2 d' (2\gamma^2 - d\alpha^2) - \alpha^2 (a^2 n + 2du_1^2) \\ &= 2b^2 d' \gamma^2 - \alpha^2 (2c^2 n + 2du_1^2) \\ &= 2((bd'\gamma)^2 - n\alpha^2 u_2^2)/d' \equiv 0 \pmod{16}, \end{aligned}$$

we may choose u_3 such that $8 \mid bd'\beta + \alpha u_3$. Then

$$\begin{aligned} \langle \Lambda, \Lambda' \rangle_2 &= [L_1 L_3(P_2), -1]_2 = [(u_1 - u_2)(bd'\beta + \alpha u_3 - 2\gamma u_2), -1]_2 \\ &= [-2\gamma u_2(u_1 - u_2), -1]_2 = [2, -1]_2 + [u_2 - u_1, -1]_2 \\ &= [\gamma, -1]_2 + [1 - u_1 \gamma, -1]_2 \\ &= \left[\frac{-1}{\gamma}\right] + \left[1 - 2\left[\frac{2}{d}\right], -1\right]_2 = \left[\frac{-1}{\gamma}\right] + \left[\frac{2}{d}\right]. \end{aligned}$$

Since $d\alpha^2 \equiv -d'\beta^2 \pmod{\gamma}$, we have $\left(\frac{-1}{\gamma}\right) = \left(\frac{n}{\gamma}\right) = \left(\frac{\gamma}{n}\right)$ and

$$\langle \Lambda, \Lambda' \rangle_2 = \left[\frac{\gamma}{n}\right] + \left[\frac{2}{d}\right].$$

For $q \mid a$, take $P_q = (1, 0, u_2, a\sqrt{n})$ where $u_2^2 = c^2 d'$. Since

$$\begin{aligned} (bd'\beta - 2\gamma u_2)(bd'\beta + 2\gamma u_2) &= b^2 d'^2 \beta^2 - 4c^2 d' \gamma^2 \\ &\equiv 2c^2 d' (d'\beta^2 - 2\gamma^2) = -2c^2 n \alpha^2 \pmod{q}, \end{aligned}$$

we may choose u_2 such that $q \mid bd'\beta + 2\gamma u_2$ if $q \mid \alpha$. If $q \nmid bd'\beta \pm 2\gamma u_2$, then $q \mid \beta$, which contradicts to the primitivity of (α, β, γ) . Therefore, $q \nmid bd'\beta - 2\gamma u_2$. If $q \nmid \alpha$, clearly we have $q \nmid bd'\beta \pm 2\gamma u_2$. Then

$$L_1 L_3(P_q) = -u_2(bd'\beta - 2\gamma u_2 + a\alpha\sqrt{n}) \in \mathbb{Z}_q^\times$$

and

$$\langle \Lambda, \Lambda' \rangle_q = [L_1 L_3(P_q), -1]_q = 0.$$

Similarly, $\langle \Lambda, \Lambda' \rangle_q = 0$ for $q \mid b$. Hence

$$\langle \Lambda, \Lambda' \rangle = \langle \Lambda, \Lambda' \rangle_2 = \left[\frac{\gamma}{n}\right] + \left[\frac{2}{d}\right].$$

Since $\mathbf{R}_n = (\mathbf{A}, \mathbf{b}_2)$, we have $\mathcal{A}[2] \cap \mathcal{A}^2 = \{[(1)], [(2d, \sqrt{-n})]\}$. Since $\mathbf{b}_2 \in \text{Im } \mathbf{A}$, we have

$$\text{Im } \mathbf{R}_n = \text{Im } \mathbf{A} = \{\mathbf{u} : \mathbf{1}^T \mathbf{u} = 0\}.$$

By [Proposition 2.2](#), $[(2d, \sqrt{-n})] \in \mathcal{A}^4$ if and only if

$$\mathbf{b}_\gamma = \left(\left[\frac{\gamma}{p_1}\right], \dots, \left[\frac{\gamma}{p_k}\right]\right)^T \in \text{Im } \mathbf{R}_n,$$

if and only if

$$\langle \Lambda, \Lambda' \rangle = \left[\frac{\gamma}{n}\right] + \left[\frac{2}{d}\right] = \mathbf{1}^T \mathbf{b}_\gamma + \left[\frac{2}{d}\right] = \left[\frac{2}{d}\right].$$

In conclusion, the Cassels pairing is non-degenerate if and only if $h_8(n) = \left[\frac{2}{d}\right]$. $\square$

5. EQUIDISTRIBUTION OF RESIDUE SYMBOLS

In this and next sections, we will prove [Theorem 1.3](#).

5.1. Residue symbols.

Definition 5.1. Denote by $I = \sqrt{-1}$ and $\mathbb{Z}[I]$ the ring of Gaussian integers.

- (1) A prime element λ of $\mathbb{Z}[I]$ is called *Gaussian* if it is not a rational prime.
- (2) An integer $\lambda \in \mathbb{Z}[I]$ is called *primary* if $\lambda \equiv 1 \pmod{2+2I}$.

Recall the quadratic and quartic residue symbols on $\mathbb{Z}[I]$, referring to [Hec81, p. 196] and [IR90]. Denote by $\mathbf{N} = \mathbf{N}_{\mathbb{Q}(I)/\mathbb{Q}}$ the norm from $\mathbb{Q}(I)$ to $\mathbb{Q}$. For any $\alpha \in \mathbb{Z}[I]$ and prime element λ prime to $1+I$, define

$$\left(\frac{\alpha}{\lambda}\right)_2 \in \{0, \pm 1\} \quad \text{such that} \quad \left(\frac{\alpha}{\lambda}\right)_2 \equiv \alpha^{\frac{\mathbf{N}\lambda-1}{2}} \pmod{\lambda}. \quad (5.1)$$

For any element λ prime to $1+I$ with a prime decomposition $\lambda = \prod_{i=1}^k \lambda_i$, define $\left(\frac{\alpha}{\lambda}\right)_2 = \prod_{i=1}^k \left(\frac{\alpha}{\lambda_i}\right)_2$.

For any $\alpha \in \mathbb{Z}[I]$ and primary prime λ , define

$$\left(\frac{\alpha}{\lambda}\right)_4 \in \{0, \pm 1, \pm I\} \quad \text{such that} \quad \left(\frac{\alpha}{\lambda}\right)_4 \equiv \alpha^{\frac{\mathbf{N}\lambda-1}{4}} \pmod{\lambda}. \quad (5.2)$$

For any primary element λ with a primary prime decomposition $\lambda = \prod_{i=1}^k \lambda_i$, define $\left(\frac{\alpha}{\lambda}\right)_4 = \prod_{i=1}^k \left(\frac{\alpha}{\lambda_i}\right)_4$. Let λ and λ' be two coprime primary primes. Then we have the quartic reciprocity law

$$\left(\frac{\lambda}{\lambda'}\right)_4 = \left(\frac{\lambda'}{\lambda}\right)_4 (-1)^{\frac{\mathbf{N}\lambda-1}{4} \cdot \frac{\mathbf{N}\lambda'-1}{4}}.$$

Certainly, $\left(\frac{\alpha}{\lambda}\right)_2 = \left(\frac{\alpha}{\lambda}\right)_4^2$.

Let p be a rational prime such that $p \equiv 1 \pmod{4}$. Let a be a rational integer such that $\left(\frac{a}{p}\right) = 1$. By abuse of notations, we define

$$\left(\frac{a}{p}\right)_4 := \left(\frac{a}{\lambda}\right)_4, \quad (5.3)$$

where λ is a primary prime such that $\mathbf{N}\lambda = p$. For any rational integer $d = p_1 \cdots p_k$ with $p_i \equiv 1 \pmod{4}$, define $\left(\frac{a}{d}\right)_4 = \prod_{i=1}^k \left(\frac{a}{p_i}\right)_4$.

5.2. Analytic results. Let F be a number field with degree n , Δ the discriminant of F and $\mathcal{O}$ the ring of integers of F . Denote by $\mathbf{N} = \mathbf{N}_{F/\mathbb{Q}}$ the norm from F to $\mathbb{Q}$.

For an ideal $\mathfrak{f}$ of $\mathcal{O}$, denote by $I(\mathfrak{f})$ the group of fractional ideals prime to $\mathfrak{f}$ and $P_{\mathfrak{f}}$ the subgroup consisting of principal fractional ideals $(\gamma) = \gamma\mathcal{O}$ with totally real $\gamma \equiv 1 \pmod{\mathfrak{f}}$. A character χ of $I(\mathfrak{f})/P_{\mathfrak{f}}$ is called a *character modulo $\mathfrak{f}$* . It can be viewed as a character on $I(\mathfrak{f})$. If $\mathfrak{a}$ is a fractional ideal not coprime to $\mathfrak{f}$, define $\chi(\mathfrak{a}) = 0$. Denote by

$$\Lambda(\mathfrak{a}) = \begin{cases} \log \mathbf{N}\mathfrak{p}, & \text{if } \mathfrak{a} = \mathfrak{p}^m \text{ with } m \geq 1; \\ 0, & \text{otherwise} \end{cases} \quad (5.4)$$

the *von Mangoldt function*. Define

$$\psi(x, \chi) = \sum_{\mathbf{Na} \leq x} \chi(\mathbf{a}) \Lambda(\mathbf{a}). \quad (5.5)$$

Denote by χ_0 the principal character on $I(\mathfrak{f})/P_{\mathfrak{f}}$.

Proposition 5.2 ([IK04, p. 112, Exercise 7]). *If $\chi \neq \chi_0$ is a character modulo $\mathfrak{f}$ and $1 \leq T \leq x$, then*

$$\psi(x, \chi) = - \sum_{|\operatorname{Im} \rho| \leq T} \frac{x^\rho - 1}{\rho} + O(T^{-1} x \log x \log(x^n \mathbf{Nf})).$$

Here ρ runs over all the zeros of $L(s, \chi)$ with $0 \leq \operatorname{Re} \rho \leq 1$.

Similar to the classical process on the estimation of $\psi(x, \chi)$ as in [Dav80, § 19], we derive the following explicit formula

$$\psi(x, \chi) = - \frac{x^{\beta'}}{\beta'} + R(x, T) \quad (5.6)$$

with

$$R(x, T) \ll x \log^2(x \mathbf{Nf}) \exp\left(-\frac{c_1 \log x}{\log(T \mathbf{Nf})}\right) + T^{-1} x \log x \cdot \log(x^n \mathbf{Nf}) + x^{\frac{1}{4}} \log x.$$

We also use the estimation on the number of zeroes in [Lan18, Satz LXXI]. Here c_1 is a positive constant and the term $-\frac{x^{\beta'}}{\beta'}$ occurs only if χ is a real character such that $L(s, \chi)$ has a zero β' satisfying

$$\beta' > 1 - \frac{c_2}{\log \mathbf{Nf}}$$

with c_2 a positive constant.

We recall the Siegel Theorem as in [Fog61, Theorem] and [Fog63, Satz].

Theorem 5.3. *Let χ be a character modulo an integral $\mathfrak{f}$ and $D = |\Delta| \mathbf{Nf} > 1$.*

(1) *There is a positive constant $c_3 = c_3(n)$ such that in the region*

$$\operatorname{Re}(s) > 1 - \frac{c_3}{\log D(2 + |\operatorname{Im} s|)} > \frac{3}{4}$$

there is no zero of $L(s, \chi)$ in the case of a complex χ . For at most one real χ' , there may be a simple zero β' of $L(s, \chi')$ in this region.

(2) *For any $\varepsilon > 0$, there exists a positive constant $c_4 = c_4(n, \varepsilon)$ such that*

$$1 - \beta' > c_4(n, \varepsilon) D^{-\varepsilon}.$$

We recall the Page Theorem as a special case of [HR95, § 3, Theorem A].

Theorem 5.4. *For any $Z \geq 2$ and a suitable constant c_5 , there is at most one real primitive character χ modulo $\mathfrak{f}$ with $\mathbf{Nf} \leq Z$ such that $L(s, \chi)$ has a real zero β satisfying*

$$\beta > 1 - \frac{c_5}{\log Z}.$$

5.3. Equidistribution of residue symbols. Recall that $abc = q_1^{t_1} \cdots q_\ell^{t_\ell}$ is the prime decomposition of abc . Let $\alpha = (\alpha_1, \dots, \alpha_k)$ be a vector with $\alpha_i \in \{1, 5, 9, 13\}$ and $\alpha_1 \cdots \alpha_k \equiv 1 \pmod{8}$. Let $\mathbf{B} = (B_{ij})_{k \times k} \in M_k(\mathbb{F}_2)$ be a symmetric matrix with rank $k - 2$ such that $\mathbf{B}\mathbf{1} = \mathbf{0}$. Then $\text{Ker } \mathbf{B} = \{\mathbf{0}, \mathbf{1}, \mathbf{d}, \mathbf{d} + \mathbf{1}\}$ for some vector $\mathbf{d} = (s_1, \dots, s_k)^T$ with $s_k = 0$.

Denote by $C_k(x, \alpha, \mathbf{B})$ the set of all $n = p_1 \cdots p_k$ satisfying

- $n \leq x$ and $p_1 < \cdots < p_k$;
- $p_i \equiv \alpha_i \pmod{16}$ for all $1 \leq i \leq k$;
- $\left[\frac{p_j}{p_i}\right] = B_{ij}$ for all $1 \leq i < j \leq k$;
- $\left(\frac{p_i}{q_j}\right) = 1$ for all $1 \leq i \leq k$ and $1 \leq j \leq \ell$;
- $\left(\frac{d'}{d}\right)_4 \left(\frac{d}{d'}\right)_4 = -1$, where $d = p_1^{s_1} \cdots p_k^{s_k}$ and $d' = n/d$,

and denote by $C'_k(x, \alpha, \mathbf{B})$ the set of all $\eta = \lambda_1 \cdots \lambda_k$ satisfying

- $\mathbf{N}\eta \leq x$ and $\mathbf{N}\lambda_1 < \cdots < \mathbf{N}\lambda_k$;
- $\lambda_i \in \mathcal{P}$ and $\mathbf{N}\lambda_i \equiv \alpha_i \pmod{16}$ for all $1 \leq i \leq k$;
- $\left[\frac{\mathbf{N}\lambda_j}{\mathbf{N}\lambda_i}\right] = B_{ij}$ for all $1 \leq i < j \leq k$;
- $\left(\frac{\mathbf{N}\lambda_i}{q_j}\right) = 1$ for all $1 \leq i \leq k$ and $1 \leq j \leq \ell$;
- $\left(\frac{\delta'}{\delta}\right)_2 = -1$, where $\delta = \lambda_1^{s_1} \cdots \lambda_k^{s_k}$ and $\delta' = \eta/\delta$.

Here, $\mathcal{P}$ is the set of primary primes in $\mathbb{Z}[I]$ with positive imaginary part.

In this section, we will give an estimation of the number of $C_k(x, \alpha, \mathbf{B})$.

Lemma 5.5. *There is a bijection*

$$\begin{aligned} C'_k(x, \alpha, \mathbf{B}) &\longrightarrow C_k(x, \alpha, \mathbf{B}), \\ \eta &\longmapsto \mathbf{N}\eta. \end{aligned}$$

Proof. For any $\eta = \lambda_1 \cdots \lambda_k \in C'_k(x, \alpha, \mathbf{B})$, denote by $p_i = \mathbf{N}\lambda_i$. By the quartic reciprocity law, we have

$$\begin{aligned} \left(\frac{p_i}{p_j}\right)_4 \left(\frac{p_j}{p_i}\right)_4 &= \left(\frac{\lambda_i \bar{\lambda}_i}{\lambda_j}\right)_4 \left(\frac{\lambda_j \bar{\lambda}_j}{\lambda_i}\right)_4 = \left(\frac{\lambda_i}{\lambda_j}\right)_4 \left(\frac{\bar{\lambda}_i}{\lambda_j}\right)_4 \left(\frac{\lambda_j}{\lambda_i}\right)_4 \left(\frac{\bar{\lambda}_j}{\lambda_i}\right)_4 \\ &= \left(\frac{\lambda_j}{\lambda_i}\right)_4 \left(\frac{\lambda_j}{\bar{\lambda}_i}\right)_4 \left(\frac{\lambda_j}{\lambda_i}\right)_4 \left(\frac{\bar{\lambda}_j}{\lambda_i}\right)_4 \\ &= \left(\frac{\lambda_j}{\lambda_i}\right)_2 \overline{\left(\frac{\bar{\lambda}_j}{\lambda_i}\right)_4} \left(\frac{\bar{\lambda}_j}{\lambda_i}\right)_4 = \left(\frac{\lambda_j}{\lambda_i}\right)_2. \end{aligned}$$

Therefore,

$$\left(\frac{d'}{d}\right)_4 \left(\frac{d}{d'}\right)_4 = \left(\frac{\delta'}{\delta}\right)_2 = -1,$$

where $d = \mathbf{N}\delta$ and $d' = \mathbf{N}\delta'$. Hence $\mathbf{N}\eta \in C_k(x, \alpha, \mathbf{B})$.

For any rational prime $p \equiv 1 \pmod{4}$, there is exactly one primary prime in $\mathcal{P}$ with norm p . This gives the surjectivity. The injectivity is trivial. $\square$

Denote by $T_k(x)$ the set of all $n = p_1 \cdots p_{k-1}$ satisfying

- $n \leq x$ and $p_1 < \cdots < p_{k-1}$;
- $p_i \equiv \alpha_i \pmod{16}$ for all $1 \leq i \leq k-1$;

- $\left[\frac{p_j}{p_i}\right] = B_{ij}$ for all $1 \leq i < j \leq k-1$;
- $\left(\frac{p_i}{q_j}\right) = 1$ for all $1 \leq i \leq k-1$ and $1 \leq j \leq \ell$,

and denote by $T'_k(x)$ the set of all $\eta = \lambda_1 \cdots \lambda_{k-1}$ satisfying

- $\mathbf{N}\eta \leq x$ and $\mathbf{N}\lambda_1 < \cdots < \mathbf{N}\lambda_{k-1}$;
- $\lambda_i \in \mathcal{P}$ and $\mathbf{N}\lambda_i \equiv \alpha_i \pmod{16}$ for all $1 \leq i \leq k-1$;
- $\left[\frac{\mathbf{N}\lambda_j}{\mathbf{N}\lambda_i}\right] = B_{ij}$ for all $1 \leq i < j \leq k-1$;
- $\left(\frac{\mathbf{N}\lambda_i}{q_j}\right) = 1$ for all $1 \leq i < k$ and $1 \leq j \leq \ell$.

The independence property of Legendre symbols in [Rho09] implies that

$$\#T_k(x) \sim 2^{-(\ell+3)(k-1) - \binom{k-1}{2}} \cdot \#C_{k-1}(x), \quad (5.7)$$

where $C_k(x)$ is the set of all positive square-free integers $n \leq x$ with exactly k prime factors.

Lemma 5.6. *There is a bijection*

$$\begin{aligned} T'_k(x) &\longrightarrow T_k(x), \\ \eta &\longmapsto \mathbf{N}\eta. \end{aligned}$$

Proof. For any rational prime $p \equiv 1 \pmod{4}$, there is exactly one primary prime in $\mathcal{P}$ with norm p . This proves the surjectivity. The injectivity is trivial. $\square$

Lemma 5.7. *Let $\chi_1, \chi_2 : G \rightarrow \mathbb{F}_2$ be two different non-trivial quadratic characters on a finite group G . Then the size of $\chi_1^{-1}(i) \cap \chi_2^{-1}(j)$ is $\#G/4$ for any $i, j \in \mathbb{F}_2$.*

Proof. Note that the assumption implies that $\#G \geq 2$. The sizes of $\chi_1^{-1}(i)$ and $\chi_2^{-1}(j)$ are $\#G/2$. Since $\chi_1 \neq \chi_2$, these two sets always have a common element, which means that $(\chi_1, \chi_2) : G \rightarrow \mathbb{F}_2^2$ is surjective. The result then follows. $\square$

Lemma 5.8. *Assume that $\pi \in \mathcal{P}$ and $p = \mathbf{N}\pi$. Then $\left(\frac{x}{\pi}\right)_2$ and $\left(\frac{\mathbf{N}x}{p}\right)$ are different non-trivial quadratic characters on $(\mathbb{Z}[I]/p\mathbb{Z}[I])^\times$.*

Proof. Since $\mathbf{N} : (\mathbb{Z}[I]/p\mathbb{Z}[I])^\times \rightarrow (\mathbb{Z}/p\mathbb{Z})^\times$ is surjective, $\left(\frac{\mathbf{N}x}{p}\right)$ is non-trivial. Let $\gamma \in \mathbb{Z}[I]$ be an element such that $\pi\gamma \equiv 1 \pmod{\bar{\pi}}$. Let $x = \bar{\pi}\bar{\gamma} + \alpha\pi\gamma$ for some $\alpha \in \mathbb{Z}$ coprime to p . Then

$$\left(\frac{x}{\pi}\right)_2 = \left(\frac{\bar{\pi}\bar{\gamma}}{\pi}\right)_2 = 1.$$

Denote by $A = (\pi\gamma)^2 + (\bar{\pi}\bar{\gamma})^2$. Then $\mathbf{N}(x) \equiv \alpha A \pmod{p}$ and

$$\left(\frac{\mathbf{N}x}{p}\right) = \left(\frac{\alpha A}{p}\right).$$

Hence $\left(\frac{x}{\pi}\right)_2 \neq \left(\frac{\mathbf{N}x}{p}\right)$ by taking $\left(\frac{\alpha}{p}\right) = -\left(\frac{A}{p}\right)$. $\square$

Lemma 5.9. *Let $\varphi(\eta)$ be the cardinality of $G = (\mathbb{Z}[I]/\mathfrak{c}_\eta)^\times$. Then*

$$\#\mathcal{A}_\eta = 2^{-k-\ell-4}\varphi(\eta).$$

Proof. By the Chinese Remainder Theorem, we have a natural isomorphism

$$G \cong \left(\frac{\mathbb{Z}[I]}{16\mathbb{Z}[I]} \right)^\times \times \prod_{i=1}^{k-1} \left(\frac{\mathbb{Z}[I]}{\mathbf{N}\lambda_i\mathbb{Z}[I]} \right)^\times \times \prod_{\mu|Q_1} \left(\frac{\mathbb{Z}[I]}{\mathbf{N}\mu\mathbb{Z}[I]} \right)^\times \times \prod_{q|Q_2} \left(\frac{\mathbb{Z}[I]}{q\mathbb{Z}[I]} \right)^\times$$

$$\beta \mapsto (\beta_0, \beta_1, \dots, \beta_{k-1}, \beta'_\mu, \beta'_q).$$

Then $\beta \in \mathcal{A}_\eta$ if and only if

- (1) $\beta_0 \equiv 1 \pmod{2+2I}$ and $\mathbf{N}\beta_0 \equiv \alpha_k \pmod{16}$;
- (2) $\left[\frac{\mathbf{N}\beta_i}{\mathbf{N}\lambda_i} \right] = B_{ik}$ for all $1 \leq i \leq k-1$;
- (3) $\left(\frac{\mathbf{N}\beta'_\mu}{\mathbf{N}\mu} \right) = 1$ for all $\mu \mid Q_1$;
- (4) $\left(\frac{\mathbf{N}\beta'_q}{q} \right) = 1$ for all $q \mid Q_2$;
- (5) $\prod_{s_i=1} \left(\frac{\beta_i}{\lambda_i} \right)_2 = -\left(\frac{\eta/\delta}{\delta} \right)_2$.

(1) selects $\frac{1}{4} \times \frac{1}{4}$ number of elements in $(\mathbb{Z}[I]/16\mathbb{Z}[I])^\times$. Note that $(\mathbb{Z}[I]/\lambda_i\mathbb{Z}[I])^\times \cong (\mathbb{Z}/\mathbf{N}\lambda_i\mathbb{Z})^\times$, each conditions in (2) to (4) select half as many elements in each corresponding component.

To treat (5), we choose $\beta_1, \dots, \beta_{k-1}$ as follows. Since $s_k = 0$, there is some $s_j = 1$ for $1 \leq j \leq k-1$. For $i = 1, 2, \dots, j-1, j+1, \dots, k-1$, we choose $\beta_i \in (\mathbb{Z}[I]/\mathbf{N}\lambda_i\mathbb{Z}[I])^\times$ satisfying (2), and there are half number of $(\mathbb{Z}[I]/\mathbf{N}\lambda_i\mathbb{Z}[I])^\times$ choices. With above chosen $\beta_1, \dots, \beta_{j-1}, \beta_{j+1}, \dots, \beta_{k-1}$, applying [Lemmas 5.7](#) and [5.8](#) to $\pi = \lambda_j$, (5) and $\left[\frac{\mathbf{N}\beta_j}{\mathbf{N}\lambda_j} \right] = B_{jk}$ selects $\frac{1}{4}$ number of elements in $(\mathbb{Z}[I]/\mathbf{N}\lambda_j\mathbb{Z}[I])^\times$. Hence

$$\frac{\#\mathcal{A}_\eta}{\varphi(\eta)} = \frac{1}{16} \times \frac{1}{2^{k-1}} \times \frac{1}{2^\ell} \times \frac{1}{2} = 2^{-k-\ell-4}. \quad \square$$

For any $\eta \in T'_k(x)$, denote by $h(\eta)$ the number of primes $\theta \in \mathcal{P}$ such that $\mathbf{N}\lambda_{k-1} < \mathbf{N}\theta \leq x/\mathbf{N}\eta$ and $\theta \pmod{\mathfrak{c}_\eta} \in \mathcal{A}_\eta$. Then we have

$$\#C'_k(x, \alpha, \mathbf{B}) = \sum_{\eta \in T'_k(x)} h(\eta). \quad (5.8)$$

Denote by

$$M_1 = (\log x)^{100} \quad \text{and} \quad M_2 = \exp\left(\frac{\log x}{(\log \log x)^{100}}\right).$$

We will use

$$\sum_{\mathbf{N}\eta \in S}^*$$

to denote a summation over $\eta \in T'_k(x)$ with $\mathbf{N}\eta \in S$.

Lemma 5.10. *We have*

$$\begin{aligned} \sum_{20 < \mathbf{N}\eta \leq M_1}^* \text{Li}(x/\mathbf{N}\eta) &= o\left(\frac{x(\log \log x)^{k-1}}{\log x}\right), \\ \sum_{M_2 < \mathbf{N}\eta \leq x^{\frac{k-1}{k}}}^* \text{Li}(x/\mathbf{N}\eta) &= o\left(\frac{x(\log \log x)^{k-1}}{\log x}\right), \\ \sum_{M_1 < \mathbf{N}\eta \leq M_2}^* \text{Li}(x/\mathbf{N}\eta) &\sim \frac{\#T'_k(x)}{k-1} \log \log x. \end{aligned}$$

Proof. The proof is similar to [CO89, Lemma 3.1]. $\square$

Theorem 5.11. *Notations as above with $k > 1$. We have*

$$\#C_k(x, \alpha, \mathbf{B}) \sim 2^{-k\ell-3k-1-\binom{k}{2}} \cdot \#C_k(x),$$

where $C_k(x)$ is the set of all positive square-free integers $n \leq x$ with exactly k prime factors.

Proof. Similar to [CO89], we consider the comparison map

$$\begin{aligned} f : C'_k(x, \alpha, \mathbf{B}) &\longrightarrow T'_k(x), \\ \lambda_1 \cdots \lambda_k &\longmapsto \lambda_1 \cdots \lambda_{k-1}. \end{aligned}$$

Let Q_1 be the product of all primary primes $\mu \in \mathcal{P}$ dividing abc , and Q_2 the product of all prime $q \mid abc$ with $q \equiv 3 \pmod{4}$. For any $\eta = \lambda_1 \cdots \lambda_{k-1} \in T'_k(x)$, denote by $\mathfrak{c}_\eta = 16\mathbf{N}(\eta Q_1)Q_2\mathbb{Z}[I]$. It's easy to see that if β satisfies

- $\mathbf{N}\beta \equiv \alpha_k \pmod{16}$;
- $\left[\frac{\mathbf{N}\beta}{\mathbf{N}\lambda_i}\right] = B_{ik}$ for all $1 \leq i \leq k-1$;
- $\left(\frac{\mathbf{N}\beta}{q_j}\right) = 1$ for all $1 \leq j \leq \ell$;
- $\left(\frac{\beta}{\delta}\right)_2 = -\left(\frac{\eta/\delta}{\delta}\right)_2$, where $\delta = \lambda_1^{s_1} \cdots \lambda_k^{s_k}$,

then so is $\beta' \equiv \beta \pmod{16\mathbf{N}(\eta Q_1)Q_2}$. Denote by

$$\mathcal{A}_\eta \subseteq (\mathbb{Z}[I]/\mathfrak{c}_\eta)^\times$$

the classes of such β . Then η lies in the image of f if and only if there exists $\theta \in \mathcal{P}$ such that $\mathbf{N}\lambda_{k-1} < \mathbf{N}\theta \leq x/\mathbf{N}\eta$ and $\theta \pmod{\mathfrak{c}_\eta} \in \mathcal{A}_\eta$ by noting that $s_k = 0$.

Denote by $\pi(x)$ the number of prime ideals in $\mathbb{Z}[I]$ with norm less than or equal x . Then the prime ideal theorem over $\mathbb{Z}[I]$ tells $\pi(x) \sim \text{Li}(x)$. Certainly, $h(\eta) \leq \pi(x/\mathbf{N}\eta)$. Then we have

$$\begin{aligned} \sum_{\mathbf{N}\eta \leq 20}^* h(\eta) &\ll \text{Li}(x), \\ \sum_{20 < \mathbf{N}\eta \leq M_1}^* h(\eta) &= o\left(\frac{x(\log \log x)^{k-1}}{\log x}\right), \\ \sum_{M_2 < \mathbf{N}\eta \leq x^{\frac{k-1}{k}}}^* h(\eta) &= o\left(\frac{x(\log \log x)^{k-1}}{\log x}\right) \end{aligned} \tag{5.9}$$

by [Lemma 5.10](#). If $\mathbf{N}\eta > x^{\frac{k-1}{k}}$, then $\mathbf{N}\lambda_{k-1} > x^{\frac{1}{k}}$ and $x/\mathbf{N}\eta < x^{\frac{1}{k}} < \mathbf{N}\lambda_{k-1}$. Therefore, $h(\eta) = 0$ and

$$\sum_{\substack{* \\ x^{\frac{k-1}{k}} < \mathbf{N}\eta \leq x}} h(\eta) = 0. \quad (5.10)$$

Denote by $\pi'(y, \mathcal{B}, \mathfrak{a})$ the number of primes $\theta \in \mathbb{Z}[I]$ such that $\mathbf{N}\theta \leq y$ and $\theta \bmod \mathfrak{a} \in \mathcal{B} \subseteq (\mathbb{Z}[I]/\mathfrak{a})^\times$. Since $\theta \in \mathcal{P}$ has positive imaginary part, we have

$$h(\eta) = \frac{1}{2} \left(\pi'(x/\mathbf{N}\eta, \mathcal{A}_\eta, \mathfrak{c}_\eta) - \pi'(\mathbf{N}\lambda_{k-1}, \mathcal{A}_\eta, \mathfrak{c}_\eta) \right) + O(\sqrt{x}).$$

Here the error term originates from $-p$ with $p \equiv 3 \pmod{4}$ rational prime, and the implicit constant is absolute. By [Eqs. \(5.8\)](#) to [\(5.10\)](#) and the facts that

$$\sum_{M_1 < \mathbf{N}\eta \leq M_2}^* \pi'(\mathbf{N}\lambda_{k-1}, \mathcal{A}_\eta, \mathfrak{c}_\eta) \ll M_2 \text{Li}(M_2) = o\left(\frac{x(\log \log x)^{k-1}}{\log x}\right)$$

and M_2 is of much small order than $x^{\frac{1}{4}}$, we obtain

$$\#C'_k(x, \alpha, B) \sim \frac{1}{2} \sum_{M_1 < \mathbf{N}\eta \leq M_2}^* \pi'(x/\mathbf{N}\eta, \mathcal{A}_\eta, \mathfrak{c}_\eta) \quad (5.11)$$

with error term $o(\#C_k(x))$.

By [\[Lan94, Theorem 6.1\]](#), we have an exact sequence

$$1 \longrightarrow \mathbb{Z}[I]^\times \longrightarrow (\mathbb{Z}[I]/\mathfrak{c}_\eta)^\times \xrightarrow{\Phi} I(\mathfrak{c}_\eta)/P_{\mathfrak{c}_\eta} \longrightarrow 1 \quad (5.12)$$

where $\Phi(\gamma) = (\gamma) \bmod P_{\mathfrak{c}_\eta}$. Denote by $\pi(y, \mathcal{B}, \mathfrak{c})$ the number of prime ideals $\mathfrak{p}$ such that $\mathbf{N}\mathfrak{p} \leq y$ and $\mathfrak{p} \bmod P_{\mathfrak{c}} \in \mathcal{B} \subseteq I(\mathfrak{c})/P_{\mathfrak{c}}$. Denote by $\mathcal{T}_\eta = \Phi(\mathcal{A}_\eta)$. Then

$$\pi'(y, \mathcal{A}_\eta, \mathfrak{c}_\eta) = \pi(y, \mathcal{T}_\eta, \mathfrak{c}_\eta) \quad \text{and} \quad \#\mathcal{A}_\eta = \#\mathcal{T}_\eta \quad (5.13)$$

by noting that every prime ideal in a class of $\mathcal{T}$ corresponds to exactly one primary prime element.

Define

$$\psi(y, \mathcal{B}, \mathfrak{c}) = \sum_{\substack{\mathbf{N}\mathfrak{a} \leq y \\ \mathfrak{a} \bmod P_{\mathfrak{c}} \in \mathcal{B}}} \Lambda(\mathfrak{a}).$$

Then we have the standard asymptotic relation $\psi(y, \mathcal{B}, \mathfrak{c}) \sim \log y \cdot \pi(y, \mathcal{B}, \mathfrak{c})$.

Therefore,

$$2 \log x \cdot \#C'_k(x, \alpha, B) \sim \sum_{M_1 < \mathbf{N}\eta \leq M_2}^* \psi(x/\mathbf{N}\eta, \mathcal{T}_\eta, \mathfrak{c}_\eta) \quad (5.14)$$

by [Eqs. \(5.11\)](#) and [\(5.13\)](#). By the orthogonality of characters and the exact sequence [Eq. \(5.12\)](#), we get

$$\psi(y, \mathcal{T}_\eta, \mathfrak{c}_\eta) = \frac{4}{\varphi(\eta)} \sum_{\chi} \psi(y, \chi) \sum_{\mathfrak{a} \bmod P_{\mathfrak{c}_\eta} \in \mathcal{T}_\eta} \overline{\chi(\mathfrak{a})},$$

where χ runs over all characters of $I(\mathfrak{c}_\eta)/P_{\mathfrak{c}_\eta}$ and

$$\psi(y, \chi) = \sum_{\mathbf{N}\mathfrak{a} \leq y} \Lambda(\mathfrak{a}) \chi(\mathfrak{a}).$$

Therefore,

$$2 \log x \cdot \#C'_k(x, \alpha, B) \sim S_1 + S_2, \quad (5.15)$$

where

$$\begin{aligned} S_1 &= \sum_{M_1 < \mathbf{N}\eta \leq M_2}^* \frac{4\#\mathcal{T}_\eta}{\varphi(\eta)} \psi(x/\mathbf{N}\eta, \chi_0), \\ S_2 &= \sum_{M_1 < \mathbf{N}\eta \leq M_2}^* \frac{4}{\varphi(\eta)} \sum_{\chi \neq \chi_0} \psi(x/\mathbf{N}\eta, \chi) \sum_{\mathbf{a} \bmod P_{\mathbf{c}_\eta} \in \mathcal{T}_\eta} \overline{\chi(\mathbf{a})}. \end{aligned}$$

The main term is

$$\begin{aligned} S_1 &= 2^{-k-\ell-2} \sum_{M_1 < \mathbf{N}\eta \leq M_2}^* \psi(x/\mathbf{N}\eta, \chi_0) && \text{by Lemma 5.9 and Eq. (5.13)} \\ &\sim 2^{-k-\ell-2} \sum_{M_1 < \mathbf{N}\eta \leq M_2}^* \log(x/\mathbf{N}\eta) \text{Li}(x/\mathbf{N}\eta) \\ &\sim 2^{-k-\ell-2} \log x \sum_{M_1 < \mathbf{N}\eta \leq M_2}^* \text{Li}(x/\mathbf{N}\eta) \\ &\sim \frac{\log x \cdot \log \log x}{(k-1) \cdot 2^{k+\ell+2}} \cdot \#T'_k(x) && \text{by Lemma 5.10} \\ &\sim \frac{\log x \cdot \log \log x}{(k-1) \cdot 2^{k\ell+3k+\binom{k}{2}}} \cdot \#C_{k-1}(x) && \text{by Lemma 5.6 and Eq. (5.7)} \\ &\sim 2^{-k\ell-3k-\binom{k}{2}} \log x \cdot \#C_k(x) && \text{by Eq. (1.1)}. \end{aligned}$$

By Eq. (5.14) and Lemma 5.5, this theorem is reduced to showing that S_2 is an error term.

Denote by $\mathfrak{f}$ the conductor of the exceptional primitive conductor with $Z = 256M_2$ in the Page Theorem 5.4. Then $S_2 = S_3 + S_4$, where

$$\begin{aligned} S_3 &= \sum_{\substack{M_1 < \mathbf{N}\eta \leq M_2 \\ \mathfrak{f} | \mathbf{c}_\eta}}^* \frac{4}{\varphi(\eta)} \sum_{\chi \neq \chi_0} \psi(x/\mathbf{N}\eta, \chi) \sum_{\mathbf{a} \bmod P_{\mathbf{c}_\eta} \in \mathcal{T}_\eta} \overline{\chi(\mathbf{a})}, \\ S_4 &= \sum_{\substack{M_1 < \mathbf{N}\eta \leq M_2 \\ \mathfrak{f} \nmid \mathbf{c}_\eta}}^* \frac{4}{\varphi(\eta)} \sum_{\chi \neq \chi_0} \psi(x/\mathbf{N}\eta, \chi) \sum_{\mathbf{a} \bmod P_{\mathbf{c}_\eta} \in \mathcal{T}_\eta} \overline{\chi(\mathbf{a})}. \end{aligned}$$

We have

$$\begin{aligned} S_3 &\ll \sum_{\substack{M_1 < \mathbf{N}\eta \leq M_2 \\ \mathfrak{f} | \mathbf{c}_\eta}}^* \psi(x/\mathbf{N}\eta, \chi_0) \ll x \sum_{\substack{M_1 < \mathbf{N}\eta \leq M_2 \\ \mathfrak{f} | \mathbf{c}_\eta}}^* (\mathbf{N}\eta)^{-1} \\ &= \frac{x}{\mathbf{N}\mathfrak{f}} \sum_{M_1 < t\mathbf{N}\mathfrak{f} \leq M_2} t^{-1} \sum_{\substack{\mathfrak{f} | \mathbf{c}_\eta \\ \mathbf{N}\eta = t\mathbf{N}\mathfrak{f}}}^* 1 \ll \frac{x \log M_2}{\mathbf{N}\mathfrak{f}}. \end{aligned}$$

By the Page [Theorem 5.4](#) for $Z = 256M_2$, there is a positive constant c_6 such that the Siegel zero β of the primitive character with modulus $\mathfrak{f}$ has the property

$$\beta > 1 - \frac{c_6}{\log 256M_2}.$$

By the Siegel [Theorem 5.3](#) for $F = \mathbb{Q}(I)$, there is a constant $c_4 = c_4(2, 1/200) > 0$ such that

$$\beta \leq 1 - c_4(4\mathbf{N}\mathfrak{f})^{-1/200}.$$

Therefore, $\mathbf{N}\mathfrak{f} \gg (\log M_2)^{100}$ and $S_3 \ll x(\log M_2)^{-99}$ is an error term.

Since there is no Siegel zero in S_4 , we can apply the explicit formula [Eq. \(5.6\)](#) with $T = (\mathbf{N}\eta)^4$ to all the $\psi(x/\mathbf{N}\eta, \chi)$ in S_4 . Then we obtain

$$\begin{aligned} \psi(x/\mathbf{N}\eta, \chi) &\ll x(\mathbf{N}\eta)^{-1}(\log x)^2 \exp\left(-\frac{c_7 \log(x/\mathbf{N}\eta)}{\log \mathbf{N}\eta}\right) \\ &\quad + x(\mathbf{N}\eta)^{-5}(\log x)^2 + x^{1/4}(\mathbf{N}\eta)^{-1/4} \log(x/\mathbf{N}\eta) \end{aligned}$$

and $S_4 \ll S_5 + S_6 + S_7$, where

$$\begin{aligned} S_5 &= \sum_{\substack{M_1 < \mathbf{N}\eta \leq M_2 \\ \mathfrak{f} \nmid \mathfrak{c}_\eta}}^* x(\mathbf{N}\eta)^{-1}(\log x)^2 \exp\left(-\frac{c_7 \log(x/\mathbf{N}\eta)}{\log \mathbf{N}\eta}\right), \\ &\ll x(\log x)^2 \exp(-c_8(\log \log x)^{100}) \cdot \sum_{\substack{M_1 < \mathbf{N}\eta \leq M_2 \\ \mathfrak{f} \nmid \mathfrak{c}_\eta}}^* (\mathbf{N}\eta)^{-1} \\ &\ll x(\log x)^3 \exp(-c_8(\log \log x)^{100}), \\ S_6 &= \sum_{\substack{M_1 < \mathbf{N}\eta \leq M_2 \\ \mathfrak{f} \nmid \mathfrak{c}_\eta}}^* x(\mathbf{N}\eta)^{-5}(\log x)^2 \ll x(\log x)^2 M_1^{-3} \ll x(\log x)^{-200}, \\ S_7 &= \sum_{\substack{M_1 < \mathbf{N}\eta \leq M_2 \\ \mathfrak{f} \nmid \mathfrak{c}_\eta}}^* x^{1/4}(\mathbf{N}\eta)^{-1/4} \log(x/\mathbf{N}\eta) \ll x^{1/4} \log x \cdot M_2^{3/4} \ll x^{1/2}. \end{aligned}$$

Hence S_4 is also an error term. This finishes the proof. $\square$

6. DISTRIBUTION RESULT

Assume that $\text{Sel}_2(E/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$. Let $n = p_1 \cdots p_k$ be an element in $\mathcal{Q}_k(x)$ with $p_1 < \cdots < p_k$. Then $n \in \mathcal{P}_k(x)$ if and only if $h_4(n) = 1$ and $h_8(n) \equiv \frac{d-1}{4} \pmod{2}$, where d is a certain divisor of n . As shown in the proof of [Theorem 1.1 \(B\)](#), the rank of $\mathbf{A} = \mathbf{A}_n$ is $k-1$ or $k-2$.

Assume that $\text{rank } \mathbf{A} = k-2$. As shown in the proof of [Theorem 1.1 \(B\)](#), $h_4(n) = 1$ if and only if $\mathbf{b}_2 \notin \text{Im } \mathbf{A}$. In this case, $d = p_1^{s_1} \cdots p_k^{s_k} \equiv 5 \pmod{8}$, where $\text{Ker } \mathbf{A} = \{\mathbf{0}, \mathbf{1}, \mathbf{d}, \mathbf{d} + \mathbf{1}\}$ and $\mathbf{d} = (s_1, \dots, s_k)^T$. We may assume that $s_k = 0$. By [\[JY11, Theorem 3.3\(ii\)\]](#), $h_8(n) = 1$ if and only if

$$\left(\frac{d}{d'}\right)_4 \left(\frac{d'}{d}\right)_4 = -1, \tag{6.1}$$

where $d' = n/d$.

Assume that $\text{rank } \mathbf{A} = k - 1$. Then $h_4(n) = 1$, $\mathbf{b}_2 \in \text{Im } \mathbf{A}$ and $d = p_1^{s_1} \cdots p_k^{s_k}$, where $\mathbf{A}\mathbf{d} = \mathbf{b}_2$ and $\mathbf{d} = (s_1, \dots, s_k)^T$. By [JY11, Theorem 3.3(iii), (iv)], $h_8(n) = 1$ if and only if

$$\left(\frac{2d}{d'}\right)_4 \left(\frac{2d'}{d}\right)_4 = (-1)^{\frac{n-1}{8}}$$

where $d' = n/d$.

Proposition 6.1 ([BCJ+06]). *Denote by $\mathcal{B}_{k,r}$ the set of $k \times k$ symmetric matrices over $\mathbb{F}_2$ with rank r . Then*

$$\#\mathcal{B}_{k,r} = u_{r+1} 2^{\binom{r+1}{2}} \cdot \prod_{i=0}^{k-r-1} \frac{2^k - 2^i}{2^{k-r} - 2^i},$$

where u_i is defined in [Theorem 1.3](#).

Proof of Theorem 1.3. For $k \geq 2$, let $\mathcal{B}$ be the set of all symmetric $\mathbf{B} \in M_k(\mathbb{F}_2)$ with rank $k - 2$ and $\mathbf{B}\mathbf{1} = \mathbf{0}$. Let $\mathcal{I}$ be the set of all vectors $\alpha = (\alpha_1, \dots, \alpha_k)$ with $\alpha_i \in \{1, 5, 9, 13\}$ and $\alpha_1 \cdots \alpha_k \equiv 1 \pmod{8}$. Denote by $\mathcal{I}_{\mathbf{B}}$ the set of all $\alpha \in \mathcal{I}$ such that $\mathbf{b}(\alpha) \notin \text{Im } \mathbf{B}$, where $\mathbf{b}(\alpha) = \left(\left[\frac{2}{\alpha_1}\right], \dots, \left[\frac{2}{\alpha_k}\right]\right)^T$. Since $\alpha_1 \cdots \alpha_k \equiv 1 \pmod{8}$, we have $\mathbf{b}(\alpha)^T \mathbf{1} = 0$. For any $\mathbf{B} \in \mathcal{B}$ and $\alpha \in \mathcal{I}_{\mathbf{B}}$, $C_k(x, \alpha, \mathbf{B})$ is the set of all $n = p_1 \cdots p_k \in \mathcal{P}_k(x)$ satisfying

- $p_1 < \cdots < p_k$ and $\mathbf{A}_n = \mathbf{B}$;
- $p_i \equiv \alpha_i \pmod{16}$ for all $1 \leq i \leq k$;
- $\left(\frac{p_i}{q_j}\right) = 1$ for all $1 \leq i \leq k$ and $1 \leq j \leq \ell$

by [Eq. \(6.1\)](#). Moreover, if $\mathbf{B} \in \mathcal{B}$ and $\alpha \notin \mathcal{I}_{\mathbf{B}}$, then $C_k(x, \alpha, \mathbf{B}) \cap \mathcal{P}_k(x) = \emptyset$. Therefore, the number $N_1(x)$ of those $n \in \mathcal{P}_k(x)$ with rank $\mathbf{A}_n = k - 2$ is

$$N_1(x) = \sum_{\mathbf{B} \in \mathcal{B}} \sum_{\alpha \in \mathcal{I}_{\mathbf{B}}} \#C_k(x, \alpha, \mathbf{B}) \sim 2^{-k\ell-3k-1-\binom{k}{2}} \cdot \#C_k(x) \cdot \sum_{\mathbf{B} \in \mathcal{B}} \#\mathcal{I}_{\mathbf{B}}$$

by [Theorem 5.11](#).

Now we count the number of $\mathcal{I}_{\mathbf{B}}$ with given $\mathbf{B}$. Given $\mathbf{b} = (b_1, \dots, b_k)^T \notin \text{Im } \mathbf{B}$ with $\mathbf{b}^T \mathbf{1} = 0$, the number of α with $\mathbf{b}(\alpha) = \mathbf{b}$ is 2^k . This is because $\alpha_i = 1, 9$ if $b_i = 0$ and $\alpha_i = 5, 13$ if $b_i = 1$. Since $\mathbf{B}$ is symmetric and $\mathbf{B}\mathbf{1} = \mathbf{0}$, the size of $\text{Im } \mathbf{B} \subset \mathcal{H}_n := \{\mathbf{u} : \mathbf{1}^T \mathbf{u} = 0\}$ is 2^{k-2} . If $\mathbf{b}^T \mathbf{1} = 0$ and $\text{rank}(\mathbf{B}, \mathbf{b}) = k - 1$, then $\mathbf{b} \in \mathcal{H}_n - \text{Im } \mathbf{B}$ has 2^{k-2} choices. Consequently, $\#\mathcal{I}_{\mathbf{B}} = 2^{2k-2}$ and then

$$N_1(x) \sim 2^{-k\ell-k-3-\binom{k}{2}} \cdot \#C_k(x) \cdot \#\mathcal{B}.$$

The left-top minor of $\mathbf{B}$ of order $k - 1$ induces a bijection $\mathcal{B} \rightarrow \mathcal{B}_{k-1, k-2}$. So $\#\mathcal{B} = \#\mathcal{B}_{k-1, k-2}$ and we get

$$N_1(x) \sim 2^{-k\ell-k-3}(1 - 2^{1-k})u_{k-1} \cdot \#C_k(x).$$

The number $N_2(x)$ of $n \in \mathcal{P}_k(x)$ with rank $\mathbf{A}_n = k - 1$ can be obtained similarly:

$$N_2(x) \sim 2^{-k-k\ell-2}u_k \cdot \#C_k(x).$$

We refer to our previous paper [Wan17] for more details. This finishes the proof of this theorem. $\square$

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SCHOOL OF MATHEMATICS AND STATISTICS, SHAANXI NORMAL UNIVERSITY, XI'AN 710119, CHINA

Email address: zhangjiewang@snnu.edu.cn

SCHOOL OF MATHEMATICS, HEFEI UNIVERSITY OF TECHNOLOGY, HEFEI, ANHUI 230000, CHINA

Email address: zhangshenxing@hfut.edu.cn