

ON NON-CONGRUENT NUMBERS AS MULTIPLES OF NON-CONGRUENT NUMBERS

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ABSTRACT. Let $n = PQ$ be a square-free positive integer, where P is a product of primes congruent to 1 mod 8, and Q is a non-congruent number with a trivial 2-primary Shafarevich-Tate group. Under certain conditions on the Legendre symbols $\left(\frac{q}{p}\right)$ for primes $p \mid P, q \mid Q$, we establish a criterion characterizing when n is non-congruent with a minimal or a second minimal 2-primary Shafarevich-Tate group. We also provide a sufficient condition for n to be non-congruent with a larger 2-primary Shafarevich-Tate group. These results involve the class groups and tame kernels of quadratic fields.

1. INTRODUCTION

1.1. Background. A square-free positive integer n is called *congruent* if it is the area of a right triangle with rational lengths. This is equivalent to say, the Mordell-Weil rank of E_n over $\mathbb{Q}$ is positive, where

$$E_n : y^2 = x^3 - n^2x$$

is the associated congruent elliptic curve. Denote by $\text{Sel}_2(E_n)$ the 2-Selmer group of E_n over $\mathbb{Q}$ and

$$s_2(n) := \dim_{\mathbb{F}_2} \left(\frac{\text{Sel}_2(E_n)}{E_n(\mathbb{Q})[2]} \right) = \dim_{\mathbb{F}_2} \text{Sel}_2(E_n) - 2$$

the *pure 2-Selmer rank*. Then

$$s_2(n) = \text{rank}_{\mathbb{Z}} E_n(\mathbb{Q}) + \dim_{\mathbb{F}_2} \text{III}(E_n)[2]$$

by the exact sequence

$$0 \rightarrow E_n(\mathbb{Q})/2E_n(\mathbb{Q}) \rightarrow \text{Sel}_2(E_n) \rightarrow \text{III}(E_n)[2] \rightarrow 0,$$

where $\text{III}(E_n)$ is the Shafarevich-Tate group of $E_n/\mathbb{Q}$.

Certainly, $s_2(n) = 0$ implies that n is non-congruent with $\text{III}(E_n)[2^\infty] = 0$. The examples of $s_2(n) = 0$ can be found in [Fen97], [Isk96] and [OZ15], which are corollaries of Monsky's formula Eq. (2.8) for $s_2(n)$. This case is fully characterized in terms of the 2-primary class groups of imaginary quadratic fields, and the full Birch-Swinnerton-Dyer conjecture holds, see [TYZ17, Theorem 1.1, Corollary 1.3] and [Smi16, Theorem 1.2].

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The examples of non-congruent n with $\text{III}(E_n)[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2$ can be found in [LT00], [OZ14], [OZ15] and [Zha23]. Denote by

$$r_{2^a}(A) = \dim_{\mathbb{F}_2} \left(\frac{2^{a-1}A}{2^a A} \right) \quad (1.1)$$

the 2^a -rank of a finite abelian group A . Denote by $h_{2^a}(m)$ the 2^a -rank of the narrow class group $\mathcal{A}_m$ of the quadratic field $\mathbb{Q}(\sqrt{m})$. Denote by $(a, b)_v$ the Hilbert symbol.

Theorem 1.1 ([Wan16, Theorem 1.1]). *Let $n = p_1 \cdots p_k \equiv 1 \pmod{8}$ be a square-free positive integer with prime factors p_i such that $p_i \equiv 1 \pmod{4}$ for all i . The following are equivalent:*

- n is non-congruent with $\text{III}(E_n)[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2$;
- $h_4(-n) = 1$ and $h_8(-n) \equiv (d-1)/4 \pmod{2}$,

where d is a positive divisor of n such that either $(d, -n)_v = 1, \forall v, d \neq 1, n$, or $(2d, -n)_v = 1, \forall v$.

Theorem 1.2 ([WZ22, Theorem 1.1]). *Let $n = p_1 \cdots p_k \equiv 1 \pmod{8}$ be a square-free positive integer with prime factors p_i such that $p_i \equiv \pm 1 \pmod{8}$ for all i . The following are equivalent:*

- n is non-congruent with $\text{III}(E_n)[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2$;
- $h_4(-n) = 1, h_8(-n) = 0$.

Theorem 1.3 ([Zha23, Theorem 5.3]). *Let $n = p_1 \cdots p_k \equiv 1 \pmod{8}$ be a square-free positive integer with prime factors p_i such that $p_i \equiv \pm 1 \pmod{8}$ for all i . The following are equivalent:*

- $2n$ is non-congruent with $\text{III}(E_{2n})[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2$;
- $h_4(-n) = 1$ and $d \equiv 9 \pmod{16}$,

where d is the unique divisor of n such that $(d, n)_v = 1, \forall v$ and $d \neq 1, d \equiv 1 \pmod{4}$.

The condition that $d \equiv 9 \pmod{16}$ is equivalent to $h_8(-n) + h_8(-2n) = 1$, see [Proposition 2.9](#). This recovers [LQ23, Theorem 1.6].

Qin in [Qin22, Theorem 1.5] proved that if $p \equiv 1 \pmod{8}$ is a prime with trivial 8-rank of the tame kernel $K_2\mathcal{O}_{\mathbb{Q}(\sqrt{p})}$, then p is non-congruent. Moreover, if the 4-rank of $K_2\mathcal{O}_{\mathbb{Q}(\sqrt{p})}$ is 1, then $\text{III}(E_p/\mathbb{Q})[2^\infty] \cong (\mathbb{Z}/4\mathbb{Z})^2$.

1.2. Main results. In this paper, we want to construct non-congruent numbers n with the form $n = PQ$, where

- P is a product of different primes $\equiv 1 \pmod{8}$,
- Q is a non-congruent number prime to P , such that $\text{III}(E_Q)[2^\infty] = 0$.

Denote the prime decomposition of n by

$$n = \gcd(2, Q)p_1 \cdots p_k q_1 \cdots q_\ell,$$

where $P = p_1 \cdots p_k, Q = \gcd(2, Q)q_1 \cdots q_\ell$. Assume that there exists two vectors

$$\mathbf{u} = (u_1, \dots, u_k)^T \in \mathbb{F}_2^k \quad \text{and} \quad \mathbf{v} = (v_1, \dots, v_\ell)^T \in \mathbb{F}_2^\ell$$

such that the Legendre symbol $\left(\frac{p_i}{q_j}\right) = (-1)^{u_i v_j}$. Denote by

$$\mathbf{U}_P = \text{diag}\{u_1, \dots, u_k\} \quad \text{and} \quad \mathbf{A}_P = (a_{ij})_{k \times k}$$

matrices defined over $\mathbb{F}_2$, such that the Hilbert symbol $(p_j, -P)_{p_i} = (-1)^{a_{ij}}$.

1.2.1. $s_2(n) = 0$.

Theorem 1.4. *Assume that $\sum_{i=1}^k u_i = 0$, $\sum_{j=1}^\ell v_j = 1$, $p_1 \equiv \dots \equiv p_k \equiv 1 \pmod{8}$ and Q is non-congruent with $\text{III}(E_Q)[2^\infty] = 0$. The following are equivalent:*

- n is non-congruent with $\text{III}(E_n)[2^\infty] = 0$;
- $\mathbf{A}_P + \mathbf{U}_P$ is invertible.

1.2.2. $s_2(n) = 2$.

Theorem 1.5. *Assume that $\sum_{i=1}^k u_i = 0$, $\sum_{j=1}^\ell v_j = 1$, $p_1 \equiv \dots \equiv p_k \equiv 1 \pmod{8}$ and Q is non-congruent with $\text{III}(E_Q)[2^\infty] = 0$. The following are equivalent:*

- n is non-congruent with $\text{III}(E_n)[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2$;
- $\text{corank}(\mathbf{A}_P + \mathbf{U}_P) = 1$ and $\left(\frac{\gamma}{d}\right) = -\left(\frac{\sqrt{2+1}}{d}\right)$,

where $d \neq 1$ is a positive divisor of P such that $(d, -P)_{p_i} = (-1)^{u_i}$, $\forall p_i \mid d$; $(d, -P)_{p_i} = 1$, $\forall p_i \mid \frac{P}{d}$, and (α, β, γ) is a primitive positive solution of $d\alpha^2 + \frac{n}{d}\beta^2 = 4\gamma^2$.

Here, a primitive positive solution of $d\alpha^2 + \frac{n}{d}\beta^2 = 4\gamma^2$ is an integer solution such that $\alpha, \beta, \gamma > 0$ and $\gcd(\alpha, \beta, \gamma) = 1$.

When $\mathbf{u} = \mathbf{0}$, we obtain the following result:

Corollary 1.6. *Assume that $\left(\frac{p_i}{q_j}\right) = 1, \forall i, j$, $p_1 \equiv \dots \equiv p_k \equiv 1 \pmod{8}$ and Q is non-congruent with $\text{III}(E_Q)[2^\infty] = 0$. The following are equivalent:*

- n is non-congruent with $\text{III}(E_n)[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2$;
- $h_4(-P) = 1$ and $\left(\frac{\gamma}{P}\right) = (-1)^{h_8(-P)}$;
- $h_4(-P) = 1$ and $\left(\frac{\gamma}{P}\right) = (-1)^{r_4(K_2\mathcal{O}_{\mathbb{Q}(\sqrt{P})})}$,

where (α, β, γ) is a primitive positive solution of $P\alpha^2 + Q\beta^2 = 4\gamma^2$.

When $\ell = 0$, we obtain the following results, which are special cases of [Theorems 1.1](#) to [1.3](#).

Corollary 1.7. *Let $n = p_1 \dots p_k$ be a square-free integer where $p_1 \equiv \dots \equiv p_k \equiv 1 \pmod{8}$.*

- (1) *The following are equivalent:*
 - n is non-congruent with $\text{III}(E_n)[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2$;
 - $h_4(-n) = 1$ and $h_8(-n) = 0$;
 - $r_4(K_2\mathcal{O}_{\mathbb{Q}(\sqrt{n})}) = 0$.
- (2) *The following are equivalent:*
 - $2n$ is non-congruent with $\text{III}(E_{2n})[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2$;
 - $h_4(-n) = 1$ and $h_8(-n) + h_8(-2n) = 1$;
 - $r_4(K_2\mathcal{O}_{\mathbb{Q}(\sqrt{-2n})}) = 0$.

1.2.3. *General case.*

Theorem 1.8. *Assume that $\left(\frac{p_i}{q_j}\right) = 1, \forall i, j$, $p_1 \equiv \dots \equiv p_k \equiv 1 \pmod{8}$ and Q is non-congruent with $\text{III}(E_Q)[2^\infty] = 0$. If there is a decomposition $P = f_1 \dots f_r$ such that*

- $h_4(-f_i) = 1, \forall i$;
- $\left(\frac{p}{p'}\right) = 1$ for any $i \neq j$ and prime factors $p \mid f_i, p' \mid f_j$;
- $\left(\frac{\gamma_i}{f_j}\right) = 1$ if $i \neq j$; $\left(\frac{\gamma_i}{f_i}\right) = (-1)^{h_8(-f_i)}$,

then n is non-congruent with $\text{III}(E_n)[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^{2r}$, where $(\alpha_i, \beta_i, \gamma_i)$ is a primitive positive solution of $f_i\alpha_i^2 + \frac{n}{f_i}\beta_i^2 = 4\gamma_i^2$.

When $\ell = 0$, we obtain the following results, where (1) is just [Wan16, Theorem 1.2].

Corollary 1.9. *Let $n = p_1 \cdots p_k$ be a square-free integer where $p_1 \equiv \cdots \equiv p_k \equiv 1 \pmod{8}$.*

- (1) *If there is a decomposition $n = f_1 \cdots f_r$ such that*
 - $h_4(-f_i) = 1, h_8(-f_i) = 0, \forall i$;
 - $h_8(-n) = r$, or $h_8(-n) = r - 1$ and $[(2, \sqrt{-n})] \notin \mathcal{A}_{-n}^4$;
 - $\left(\frac{p}{p'}\right) = 1$ for any $i \neq j$ and prime factors $p \mid f_i, p' \mid f_j$,*then n is non-congruent with $\text{III}(E_n)[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^{2r}$.*
- (2) *If there is a decomposition $n = f_1 \cdots f_r$ such that*
 - $h_4(-f_i) = 1, h_8(-f_i) = 0, \forall i$;
 - $h_8(-2n) = r$;
 - $\left(\frac{p}{p'}\right) = 1$ for any $i \neq j$ and prime factors $p \mid f_i, p' \mid f_j$,*then $2n$ is non-congruent with $\text{III}(E_{2n})[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^{2r}$.*

Let's sketch the proof of these results. Since the congruent elliptic curve E_n has full rational 2-torsion, the pure 2-Selmer group $\text{Sel}'_2(E_n) := \text{Sel}_2(E_n)/E_n(\mathbb{Q})[2]$ can be identified with a set of triples $(d_1, d_2, d_3) \in (\mathbb{Q}^\times/\mathbb{Q}^{\times 2})^3$, where d_1, d_2, d_3 may be taken as square-free integers. The local conditions for Selmer elements translate into certain quadratic residue conditions, which in turn correspond to the 4-ranks of class groups of associated quadratic fields. As established in [Wan16], $E_n(\mathbb{Q})$ is finite with $\text{III}(E_n)[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^{s_2(n)}$ if and only if the Cassels pairing on $\text{Sel}'_2(E_n)$ is non-degenerate. This condition can be expressed in terms of the 8-ranks of class groups and the 4-ranks of tame kernels of associated quadratic fields.

1.3. Notations. Denote by

- $\gcd(m, n)$ the greatest common divisor of integers m, n , where $m \neq 0$ or $n \neq 0$;
- $(a, b)_v$ the Hilbert symbol;
- $[a, b]_v$ the additive Hilbert symbol, i.e., the image of $(a, b)_v$ under the isomorphism $\{\pm 1\} \xrightarrow{\sim} \mathbb{F}_2$;
- $\left(\frac{a}{b}\right) = \prod_{p \mid b} (a, b)_p$ the Jacobi symbol, where $\gcd(a, b) = 1$ and $b > 0$;
- $\left[\frac{a}{b}\right]$ the additive Jacobi symbol, i.e., the image of $\left(\frac{a}{b}\right)$ under the isomorphism $\{\pm 1\} \xrightarrow{\sim} \mathbb{F}_2$;
- v_p the normalized valuation on $\mathbb{Q}_p$;
- $\mathbf{0} = (0, \dots, 0)^T$ and $\mathbf{1} = (1, \dots, 1)^T$;
- $r_{2^a}(A)$ the 2^a -rank of a finite abelian group A , see Eq. (1.1);

If n is a square-free positive integer, then we denote by

- $E_n : y^2 = x^3 - n^2x$ the congruent elliptic curve associated to n ;
- $\text{Sel}_2(E_n)$ the 2-Selmer group of $E_n/\mathbb{Q}$;
- $\text{III}(E_n)$ the Shafarevich-Tate group of $E_n/\mathbb{Q}$;
- $\text{Sel}'_2(E_n) := \text{Sel}_2(E_n)/E_n(\mathbb{Q})[2]$ the pure 2-Selmer group of $E_n/\mathbb{Q}$;
- $s_2(n) = \dim_{\mathbb{F}_2} \text{Sel}'_2(E_n)$ the pure 2-Selmer rank of E_n .

If n is odd with a fixed ordered prime decomposition $n = p_1 \cdots p_k$, then we denote by

- $\mathbf{A}_n = ([p_j, -n]_{p_i})_{k \times k}$ a matrix associated to n , see Eq. (2.2);
- $\mathbf{D}_{n,\varepsilon} = \text{diag}\left\{\left[\frac{\varepsilon}{p_1}\right], \dots, \left[\frac{\varepsilon}{p_k}\right]\right\}$ a matrix associated to n and ε , see Eq. (2.3);
- $\mathbf{b}_{n,\varepsilon} = \mathbf{D}_{n,\varepsilon} \mathbf{1} = \left(\left[\frac{\varepsilon}{p_1}\right], \dots, \left[\frac{\varepsilon}{p_k}\right]\right)^T$;
- $\mathbf{M}_n$ (resp. $\mathbf{M}_{2n}$) the Monsky matrix of E_n (resp. E_{2n}), see Eqs. (2.4) and (2.6);
- $\psi_n(d) = (v_{p_1}(d), \dots, v_{p_k}(d))^T$ a vector over $\mathbb{F}_2$ associated to $0 < d \mid n$.

If $m \neq 0, 1$ is a square-free integer, then we denote by

- $F_m = \mathbb{Q}(\sqrt{m})$ a quadratic field;
- $\mathbf{R}_m$ the Rédei matrix of F_m , with a submatrix $\mathbf{R}'_m$, see Eqs. (2.9) and (2.12);
- $\mathcal{A}_m$ the narrow class group of F_m ;
- D_m the discriminant of F_m ;
- $\omega_m = (D_m + \sqrt{D_m})/2$;
- $\mathcal{O}_m = \mathbb{Z} + \mathbb{Z}\omega_m$ the ring of integers of F_m ;
- $\mathcal{D}_m$ the set of all square-free positive divisors of D_m ;
- $\theta_m : \mathcal{D}_m \rightarrow \mathcal{A}_m[2]$ a two-to-one onto homomorphism, see Proposition 2.2;
- $h_{2^a}(m)$ the 2^a -rank of $\mathcal{A}_m$;
- $K_2\mathcal{O}_m$ the tame kernel of F_m ;
- $\mathbf{B}_m = \mathbf{A}_n + \mathbf{D}_{n,m/n}$ a matrix associated to m , where n is the odd part of $|m|$.

2. PRELIMINARIES

2.1. The Monsky matrix. By the 2-descent method, Monsky in [HB94, Appendix] represented the pure 2-Selmer group

$$\text{Sel}'_2(E_n) := \frac{\text{Sel}_2(E_n)}{E_n(\mathbb{Q})[2]}$$

as the kernel of a matrix $\mathbf{M}_n$ over $\mathbb{F}_2$. Let's recall it roughly. One can identify $\text{Sel}_2(E_n)$ with

$$\{\Lambda = (d_1, d_2, d_3) \in (\mathbb{Q}^\times / \mathbb{Q}^{\times 2})^3 : D_\Lambda(\mathbb{A}_\mathbb{Q}) \neq \emptyset, d_1 d_2 d_3 \equiv 1 \pmod{\mathbb{Q}^{\times 2}}\},$$

where D_Λ is a genus one curve defined by

$$\begin{cases} H_1 : -nt^2 + d_2 u_2^2 - d_3 u_3^2 = 0, \\ H_2 : -nt^2 + d_3 u_3^2 - d_1 u_1^2 = 0, \\ H_3 : 2nt^2 + d_1 u_1^2 - d_2 u_2^2 = 0. \end{cases} \quad (2.1)$$

Under this identification, $O, (n, 0), (-n, 0), (0, 0)$ and other point $(x, y) \in E_n(\mathbb{Q})$ correspond to $(1, 1, 1), (2, 2n, n), (-2n, 2, -n), (-n, n, -1)$ and $(x - n, x + n, x)$ respectively.

Let n be an odd positive square-free integer with an ordered prime decomposition $n = p_1 \cdots p_k$. Denote by

$$\mathbf{A}_{2n} = \mathbf{A}_n := (a_{ij})_{k \times k} \quad \text{where} \quad a_{ij} = [p_j, -n]_{p_i} = \begin{cases} \left[\frac{p_i}{p_i}\right], & i \neq j; \\ \left[\frac{n/p_i}{p_i}\right], & i = j, \end{cases} \quad (2.2)$$

and

$$\mathbf{D}_{n,\varepsilon} := \text{diag}\left\{\left[\frac{\varepsilon}{p_1}\right], \dots, \left[\frac{\varepsilon}{p_k}\right]\right\}. \quad (2.3)$$

Then $\mathbf{A}_n \mathbf{1} = \mathbf{0}$ and $\text{corank } \mathbf{A}_n \geq 1$.

Monsky showed that each element in $\text{Sel}'_2(E_n)$ can be represented as (d_1, d_2, d_3) , where d_1, d_2, d_3 are all positive divisors of n . The system D_Λ is locally solvable everywhere if and only if certain conditions on the Hilbert symbols hold. Then we can express $\text{Sel}'_2(E_n)$ as the kernel of the *Monsky matrix*

$$\mathbf{M}_n := \begin{pmatrix} \mathbf{A}_n + \mathbf{D}_{n,2} & \mathbf{D}_{n,2} \\ \mathbf{D}_{n,2} & \mathbf{A}_n + \mathbf{D}_{n,-2} \end{pmatrix} \quad (2.4)$$

via the isomorphism

$$\begin{aligned} \text{Sel}'_2(E_n) &\rightarrow \text{Ker } \mathbf{M}_n \\ (d_1, d_2, d_3) &\mapsto \begin{pmatrix} \psi_n(d_2) \\ \psi_n(d_1) \end{pmatrix}, \end{aligned} \quad (2.5)$$

where $\psi_n(d) := (v_{p_1}(d), \dots, v_{p_k}(d))^T \in \mathbb{F}_2^k$ for any positive divisor d of n .

Similarly, each element in $\text{Sel}'_2(E_{2n})$ can be represented as (d_1, d_2, d_3) , where d_1, d_2, d_3 are all divisors of n and $d_2 > 0, d_3 \equiv 1 \pmod{4}$. Then we can express $\text{Sel}'_2(E_{2n})$ as the kernel of the *Monsky matrix*

$$\mathbf{M}_{2n} := \begin{pmatrix} \mathbf{A}_n^T + \mathbf{D}_{n,2} & \mathbf{D}_{n,-1} \\ \mathbf{D}_{n,2} & \mathbf{A}_n + \mathbf{D}_{n,2} \end{pmatrix} \quad (2.6)$$

via the isomorphism

$$\begin{aligned} \text{Sel}'_2(E_{2n}) &\rightarrow \text{Ker } \mathbf{M}_{2n} \\ (d_1, d_2, d_3) &\mapsto \begin{pmatrix} \psi_n(|d_3|) \\ \psi_n(d_2) \end{pmatrix}. \end{aligned} \quad (2.7)$$

In both cases, we have

$$s_2(n) := \dim_{\mathbb{F}_2} \text{Sel}'_2(E_n) = \text{corank } \mathbf{M}_n. \quad (2.8)$$

2.2. The Cassels pairing. Cassels in [Cas98] defined a (skew-)symmetric bilinear pairing $\langle -, - \rangle$ on the $\mathbb{F}_2$ -vector space $\text{Sel}'_2(E_n)$. For any $\Lambda \in \text{Sel}_2(E_n)$, the equation H_i in Eq. (2.1) is locally solvable everywhere. Thus H_i is solvable over $\mathbb{Q}$ by the Hasse-Minkowski principal. Choose $Q_i \in H_i(\mathbb{Q})$ and let L_i be a linear form such that $L_i = 0$ defines the tangent plane of H_i at Q_i . For any $\Lambda' = (d'_1, d'_2, d'_3) \in \text{Sel}_2(E_n)$, define the *Cassels pairing*

$$\langle \Lambda, \Lambda' \rangle = \sum_v \langle \Lambda, \Lambda' \rangle_v \in \mathbb{F}_2 \quad \text{where} \quad \langle \Lambda, \Lambda' \rangle_v = \sum_{i=1}^3 [L_i(P_v), d'_i]_v,$$

where $P_v \in D_\Lambda(\mathbb{Q}_v)$ for each place v of $\mathbb{Q}$. This pairing is independent of the choice of P_v, Q_i and the representative Λ . It is (skew-)symmetric and satisfies $\langle \Lambda, \Lambda \rangle = 0$.

Lemma 2.1 ([Cas98, Lemma 7.2]). *The local Cassels pairing $\langle -, - \rangle_v = 0$ if*

- $v \nmid 2\infty$,
- the coefficients of H_i and L_i are all integral at v for $i = 1, 2, 3$, and
- modulo D_Λ and $L_i = 0$ by v , they define a curve of genus 1 over $\mathbb{F}_v$ together with tangents to it.

2.3. The narrow class group. Let $F_m = \mathbb{Q}(\sqrt{m})$ be a quadratic field, where $m \neq 0, 1$ is a square-free integer. We will use the notations introduced in [Section 1.3](#). Denote by $\mathbf{N} = \mathbf{N}_{F_m/\mathbb{Q}}$ the norm map. Fix an ordered decomposition of the odd part n of $|m|$: $n = p_1 \cdots p_k$. If $2 \mid D$, denote by $p_{k+1} = 2$. Let t be the number of prime factors of D_m . Then the Gauss genus theory tells:

Proposition 2.2 ([\[Hec81, Chapter 7\]](#)). (1) *The map $\theta_m : \mathcal{D}_m \rightarrow \mathcal{A}_m[2]$ defined as*

$$\theta_m(d) = [(d, \omega_m)]$$

is a two-to-one onto homomorphism. In particular,

$$h_2(m) = \dim_{\mathbb{F}_2} \mathcal{A}_m[2] = t - 1.$$

(2) *Let $\mathfrak{a}$ be a non-zero fractional ideal of F_m . Then the ideal class $[\mathfrak{a}] \in \mathcal{A}_m^2$ if and only if $\mathbf{N}\mathfrak{a} \in \mathbf{N}F_m$.*

When $m < 0$, the kernel of θ_m is $\{1, |m|\}$.

To calculate $h_4(m)$, we need the Rédei matrix, which is defined as

$$\mathbf{R}_m = ([p_j, m]_{p_i})_{t \times t}. \quad (2.9)$$

Example 2.3. *Let $n = p_1 \cdots p_k$ be an odd positive square-free integer. Denote by*

$$\mathbf{b}_{n,\varepsilon} := \left(\left[\frac{\varepsilon}{p_1} \right], \dots, \left[\frac{\varepsilon}{p_k} \right] \right)^T = \mathbf{D}_{n,\varepsilon} \mathbf{1}.$$

When $n \equiv 1 \pmod{4}$, we have

$$\begin{aligned} \mathbf{R}_n &= \mathbf{A}_n + \mathbf{D}_{n,-1}, & \mathbf{R}_{-n} &= \begin{pmatrix} \mathbf{A}_n & \mathbf{b}_{n,2} \\ \mathbf{b}_{n,-1}^T & \left[\frac{2}{n} \right] \end{pmatrix}, \\ \mathbf{R}_{2n} &= \begin{pmatrix} \mathbf{A}_n + \mathbf{D}_{n,-2} & \mathbf{b}_{n,2} \\ \mathbf{b}_{n,2}^T & \left[\frac{2}{n} \right] \end{pmatrix}, & \mathbf{R}_{-2n} &= \begin{pmatrix} \mathbf{A}_n + \mathbf{D}_{n,2} & \mathbf{b}_{n,2} \\ \mathbf{b}_{n,-2}^T & \left[\frac{2}{n} \right] \end{pmatrix}. \end{aligned}$$

When $n \equiv -1 \pmod{4}$, we have

$$\begin{aligned} \mathbf{R}_n &= \begin{pmatrix} \mathbf{A}_n + \mathbf{D}_{n,-1} & \mathbf{b}_{n,2} \\ \mathbf{b}_{n,-1}^T & \left[\frac{2}{n} \right] \end{pmatrix}, & \mathbf{R}_{-n} &= \mathbf{A}_n, \\ \mathbf{R}_{2n} &= \begin{pmatrix} \mathbf{A}_n + \mathbf{D}_{n,-2} & \mathbf{b}_{n,2} \\ \mathbf{b}_{n,-2}^T & \left[\frac{2}{n} \right] \end{pmatrix}, & \mathbf{R}_{-2n} &= \begin{pmatrix} \mathbf{A}_n + \mathbf{D}_{n,2} & \mathbf{b}_{n,2} \\ \mathbf{b}_{n,2}^T & \left[\frac{2}{n} \right] \end{pmatrix}. \end{aligned}$$

One can see that the following are equivalent:

- $d \in \mathcal{D}_m \cap \mathbf{N}F_m$;
- $X^2 - mY^2 = dZ^2$ is solvable over $\mathbb{Q}$;
- the Hilbert symbols $(d, m)_v = 1, \forall v$;
- $\mathbf{R}_m \mathbf{d} = \mathbf{0}$, where $\mathbf{d} = (v_{p_1}(d), \dots, v_{p_t}(d))^T$.

Rédei showed that θ_m induces a two-to-one onto homomorphism

$$\theta_m : \mathcal{D}_m \cap \mathbf{N}F_m \rightarrow \mathcal{A}_m[2] \cap \mathcal{A}_m^2, \quad (2.10)$$

which induces that

$$h_4(m) = \text{corank } \mathbf{R}_m - 1. \quad (2.11)$$

Denote by

$$\mathbf{R}'_m = ([p_j, m]_{p_i})_{k \times t}. \quad (2.12)$$

If $2 \mid D_m$, then $\mathbf{R}'_m$ is the submatrix of $\mathbf{R}_m$ by removing the last row; otherwise $\mathbf{R}'_m = \mathbf{R}_m$. Since $\mathbf{1}^T \mathbf{R}_m = \mathbf{0}^T$, we have

$$\text{rank } \mathbf{R}'_m = \text{rank } \mathbf{R}_m. \quad (2.13)$$

See [Rè34] and [LY20, Example 2.6].

The 8-rank $h_8(m)$ can be obtained by the following proposition, which is similar to [Wan16, Proposition 3.6]. See also [JY11, Lu15].

Proposition 2.4. *For any $d \in \mathcal{D}_m \cap \mathbf{N}F_m$, let (α, β, γ) be a primitive positive solution of*

$$d\alpha^2 - \frac{m}{d}\beta^2 = 4\gamma^2.$$

Then

- (1) $\theta_m(d) \in \mathcal{A}_m^4$ if and only if $([\gamma, m]_{p_1}, \dots, [\gamma, m]_{p_t})^T \in \text{Im } \mathbf{R}_m$;
- (2) $\sum_{i=1}^t [\gamma, m]_{p_i} = 0$.

In particular, $\theta_m(d) \in \mathcal{A}_m^4$ if and only if $\mathbf{b}_{n,\gamma} \in \text{Im } \mathbf{R}'_m$, where n is the odd part of $|m|$.

Proof. Denote by σ the non-trivial automorphism of $\mathbb{Q}(\sqrt{m})$. If p is an odd prime factor of γ , then $p \nmid m$ and $\left(\frac{m}{p}\right) = 1$. Thus $(p) = \mathfrak{p}\mathfrak{p}^\sigma$ is split in F_m and $[\gamma, m]_p = 0$. We will show that $x = (d\alpha + \beta\sqrt{m})/2 \in \mathcal{O}_m$.

- If d is odd and m is even, then both of α and β are even and $x \in \mathcal{O}_m$.
- If d, m are odd, then α and β have same parities. If moreover both of α and β are odd, then $4 \mid (d - m/d)$, $m \equiv 1 \pmod{4}$ and $x \in \mathcal{O}_m$.
- If d is even, then β is even and $x \in \mathcal{O}_m$.

Certainly, x is totally positive and $p \mid d\gamma^2 = \mathbf{N}(x)$. If both $\mathfrak{p}, \mathfrak{p}^\sigma$ divide $x\mathcal{O}_m$, then $p\mathcal{O}_m \mid x\mathcal{O}_m$ and $p \mid \alpha, \beta, \gamma$, which contradicts to $\gcd(\alpha, \beta, \gamma) = 1$. Hence only one of $\mathfrak{p}$ and $\mathfrak{p}^\sigma$ divides $x\mathcal{O}_m$. We may assume that $\mathfrak{p}^\sigma \mid x\mathcal{O}_m$ for each odd $p \mid \gamma$.

Assume that d is odd. If γ is odd, we have

$$x\mathcal{O}_m = \mathfrak{d} \prod_{p \mid \gamma} (\mathfrak{p}^\sigma)^{2v_p(\gamma)} = \gamma^2 \mathfrak{d} \mathfrak{c}^{-2}, \quad \text{where } \mathfrak{c} := \prod_{p \mid \gamma} \mathfrak{p}^{v_p(\gamma)} \text{ with } \mathbf{N}\mathfrak{c} = \gamma \quad (2.14)$$

and $\mathfrak{d} = (d, \omega_m)$. If γ is even, one can show that m is odd. Then both of α and β are odd, $8 \mid (d - m/d)$ and $m \equiv 1 \pmod{8}$. Thus $2\mathcal{O}_m = \mathfrak{q}\mathfrak{q}^\sigma$ is split in F . Similarly, only one of $\mathfrak{q}$ and $\mathfrak{q}^\sigma$ divides $x\mathcal{O}_m$. We may assume that $\mathfrak{q}^\sigma \mid x\mathcal{O}_m$. Hence we also have Eq. (2.14), where $\mathfrak{p}$ is $\mathfrak{q}$ for $p = 2$.

Assume that d is even. Then D_m is even, $m \not\equiv 1 \pmod{4}$ and $2\mathcal{O}_m = \mathfrak{q}^2$ is ramified in F . Similarly, we have Eq. (2.14), where $\mathfrak{p} = \mathfrak{p}^\sigma = \mathfrak{q}$ for $p = 2$.

- (1) By Eq. (2.14), we have $[\mathfrak{d}] = [\mathfrak{c}]^2$. Clearly, $[\mathfrak{d}] \in \mathcal{A}_m^4$ if and only if $[\mathfrak{c}] + [(a, \omega_m)] \in \mathcal{A}_m^2$ for some $a \in \mathcal{D}_m$. This is equivalent to $a\mathbf{N}\mathfrak{c} = a\gamma \in \mathbf{N}F_m$ by Proposition 2.2. Note that

- $[a\gamma, m]_p = 1$ for any odd prime $p \mid \gamma$;
- $[a\gamma, m]_\infty = 1$ because $a\gamma > 0$;
- if $2 \nmid D_m$ and γ is odd, then a is odd and $m \equiv 1 \pmod{4}$; if $2 \nmid D_m$ and γ is even, then $m \equiv 1 \pmod{8}$.

In other words, $[a\gamma, m]_v = 1$ for all $v \nmid D_m$. Thus $a\gamma \in \mathbf{N}F_m$ if and only if $[a, m]_{p_i} = [\gamma, m]_{p_i}$ for all $p_i \mid D_m$, if and only if

$$\mathbf{R}_m(v_{p_1}(a), \dots, v_{p_t}(a))^T = ([\gamma, m]_{p_1}, \dots, [\gamma, m]_{p_t})^T.$$

(2) Denote by γ_0 the odd part of γ . If $m \not\equiv 1 \pmod{4}$, then D_m is even and

$$\sum_{i=1}^t [\gamma, m]_{p_i} = \sum_{p|\gamma_0} [\gamma, m]_p = 0.$$

Here, $[\gamma, m]_\infty = 0$ because $\gamma > 0$. If $m \equiv 1 \pmod{4}$ and γ is odd, then $[\gamma, m]_2 = 0$; if $m \equiv 1 \pmod{4}$ and γ is even, then $m \equiv 1 \pmod{8}$ and $[\gamma, m]_2 = 0$, as shown in the proof of (1). Therefore

$$\sum_{i=1}^t [\gamma, m]_{p_i} = \sum_{p|\gamma_0} [\gamma_0, m]_p + [\gamma, m]_2 = 0. \quad \square$$

2.4. The tame kernel. Denote by $K_2\mathcal{O}_m$ the tame kernel of F_m . We list the results about 2-rank and 4-rank of $K_2\mathcal{O}_m$ that we will use. Assume that $|m| > 2$.

Theorem 2.5 ([BS82]). *The subgroup $K_2\mathcal{O}_m[2]$ is generated by the Steinberg symbols*

- $\{-1, d\}, d \mid m$;
- $\{-1, u + \sqrt{m}\}$, where $m = u^2 - cw^2$ for some $c = -1, \pm 2$ and $u, w \in \mathbb{N}$.

Denote by k the number of odd prime factors of m . Then

$$r_2(K_2\mathcal{O}_m) = \begin{cases} k + \log_2 \#(\{\pm 1, \pm 2\} \cap \mathbf{N}F_m); & \text{if } m > 2; \\ k - 1 + \log_2 \#(\{1, 2\} \cap \mathbf{N}F_m); & \text{if } m < -2. \end{cases}$$

Theorem 2.6 ([Qin95b, Theorem 3.4]). *Suppose that $m > 2$. Denote by V_1 the set of positive $d \mid n$ satisfying: there exists $\varepsilon \in \{\pm 1, \pm 2\}$ such that $(d, -m)_p = \left(\frac{\varepsilon}{p}\right), \forall p \mid n$. If $2 \in \mathbf{N}F_m$, then write $m = 2\mu^2 - \lambda^2, \mu, \lambda \in \mathbb{N}$ and denote by V_2 the set of positive $d \mid n$ satisfying: there exists $\varepsilon \in \{\pm 1\}$ such that $(d, -m)_p = \left(\frac{\varepsilon\mu}{p}\right), \forall p \mid n$. We have*

$$2^{r_4(K_2\mathcal{O}_m)+1} = \#V_1 + \#V_2.$$

Theorem 2.7 ([Qin95a, Theorem 4.1]). *Suppose that $m < -2$. Denote by V_1 the set of $d \mid n$ satisfying: there exists $\varepsilon \in \{1, 2\}$ such that $(d, -m)_p = \left(\frac{\varepsilon}{p}\right), \forall p \mid n$. If $2 \in \mathbf{N}F_m$, then write $m = 2\mu^2 - \lambda^2, \mu, \lambda \in \mathbb{N}$ and denote by V_2 the set of $d \mid n$ satisfying: $(d, -m)_p = \left(\frac{\mu}{p}\right), \forall p \mid n$. We have*

$$2^{r_4(K_2\mathcal{O}_m)+2} = \#V_1 + \#V_2.$$

Here, $V_2 = \emptyset$ if $2 \notin \mathbf{N}F_m$.

Let's translate these results into the language of matrices. Denote by n the odd part of $|m|$ and denote by $\mathbf{B}_m = \mathbf{A}_n + \mathbf{D}_{n,m/n}$, where $\mathbf{A}_n$ is defined as Eq. (2.2). If $m > 2$, then

$$\#\{\mathbf{x} : \mathbf{B}_m \mathbf{x} = \mathbf{b}_{n,\pm 1}, \mathbf{b}_{n,\pm 2}\} + \#\{\mathbf{x} : \mathbf{B}_m \mathbf{x} = \mathbf{b}_{n,\pm \mu}\} = 2^{r_4(K_2\mathcal{O}_m)+1}. \quad (2.15)$$

If $m < -2$, then

$$\begin{aligned} & \#\{\mathbf{x} : \mathbf{B}_m \mathbf{x} = \mathbf{0}, \mathbf{b}_{n,2}\} + \#\{\mathbf{x} : \mathbf{B}_m \mathbf{x} = \mathbf{b}_{n,\mu}\} \\ &= \begin{cases} 2^{r_4(K_2\mathcal{O}_m)+2}, & \text{if } \mathbf{b}_{n,-1} \notin \text{Im } \mathbf{B}_m; \\ 2^{r_4(K_2\mathcal{O}_m)+1}, & \text{if } \mathbf{b}_{n,-1} \in \text{Im } \mathbf{B}_m. \end{cases} \end{aligned} \quad (2.16)$$

Theorem 2.8. Assume that $n = p_1 \cdots p_k$ is an odd positive square-free integer, where all prime factors p_i are congruent to ± 1 modulo 8 and $n \equiv 1 \pmod{8}$. Write $n = \lambda^2 - 2\mu^2$ where $\lambda, \mu \in \mathbb{N}$.

- (1) We have $h_4(n) + 1 = h_4(2n) = h_4(-n) = h_4(-2n) = \text{corank } \mathbf{A}_n$.
- (2) If $h_4(-n) = 1$, then $h_8(-n) = 1 - \left\lfloor \frac{\lambda+\mu}{d} \right\rfloor$. If moreover all $p_i \equiv 1 \pmod{8}$, then $h_8(-n) = 1 - \left\lfloor \frac{\sqrt{2}+1}{n} \right\rfloor$.
- (3) If $h_4(-2n) = 1$, then $h_8(-2n) = 1 - \left\lfloor \frac{\lambda}{d} \right\rfloor$. If moreover all $p_i \equiv 1 \pmod{8}$, then $h_8(-2n) = 1 - \left\lfloor \frac{\sqrt{2}}{n} \right\rfloor$.
- (4) Assume that all $p_i \equiv 1 \pmod{8}$. We have $r_4(K_2\mathcal{O}_{-2n}) = 0$ if and only if $h_4(-n) = 1, h_8(-n) + h_8(-2n) = 1$. If $h_4(-n) = 1$, then $r_4(K_2\mathcal{O}_{-2n}) \leq 1$.
- (5) Assume that all $p_i \equiv 1 \pmod{8}$. We have $r_4(K_2\mathcal{O}_n) = 0$ if and only if $h_4(-n) = 1, h_8(-n) = 0$. If $h_4(-n) = 1$, then $r_4(K_2\mathcal{O}_n) \leq 1$.

Here, $1 < d \mid n$ such that $\mathbf{A}_n^T \psi_n(d) = \mathbf{0}$.

Proof. (1) By the quadratic reciprocity law, we have

$$\mathbf{A}_n^T = \mathbf{A}_n + \mathbf{D}_{n,-1} + \mathbf{b}_{n,-1} \mathbf{b}_{n,-1}^T. \quad (2.17)$$

By $\mathbf{b}_{n,-1}^T \mathbf{b}_{n,-1} = \mathbf{b}_{n,-1}^T \mathbf{1} = \left\lfloor \frac{-1}{n} \right\rfloor = 0$, one can show that

$$\mathbf{A}_n^T (\mathbf{I} + \mathbf{1} \mathbf{b}_{n,-1}^T) = \mathbf{A}_n + \mathbf{D}_{n,-1},$$

where $\mathbf{I} + \mathbf{1} \mathbf{b}_{n,-1}^T$ is invertible since $(\mathbf{I} + \mathbf{1} \mathbf{b}_{n,-1}^T)^2 = \mathbf{I}$. Thus

$$\text{rank } \mathbf{R}_n = \text{rank } \mathbf{R}'_{-n} = \text{rank } \mathbf{R}'_{\pm 2n} = \text{rank } \mathbf{A}_n,$$

which concludes the result by Eqs. (2.11) and (2.13).

- (2) Since $\theta_{-n}(n) = [(\sqrt{-n})]$ is the trivial class, we have

$$\mathcal{A}_{-n}[2] \cap \mathcal{A}_{-n}^2 = \{[(1)], \theta_{-n}(2)\},$$

where $\theta_{-n}(2) = \theta_{-n}(2n)$. Note that $(\lambda + 2\mu, 2, \lambda + \mu)$ is a primitive positive solution of $2\alpha^2 + \frac{n}{2}\beta^2 = 4\gamma^2$. Since $\text{Im } \mathbf{R}'_{-n} = \{\mathbf{x} : \psi(d)^T \mathbf{x} = 0\}$, by Proposition 2.4, we have $h_8(-n) = 1$ if and only if $\mathbf{b}_{n,\lambda+\mu} \in \text{Im } \mathbf{R}'_{-n}$, if and only if $0 = \psi(d)^T \mathbf{b}_{n,\lambda+\mu} = \left\lfloor \frac{\lambda+\mu}{d} \right\rfloor$.

If all $p_i \equiv 1 \pmod{8}$, then $d = n$ since $\mathbf{A}_n^T \mathbf{1} = \mathbf{0}$. Let μ' be the odd part of μ . Then

$$\left\lfloor \frac{\mu}{n} \right\rfloor = \left\lfloor \frac{n}{\mu'} \right\rfloor = \left\lfloor \frac{\lambda^2 - 2\mu^2}{\mu'} \right\rfloor = 0. \quad (2.18)$$

Since $\lambda \equiv \pm \sqrt{2}\mu \pmod{p_i}$, we have $\left\lfloor \frac{\lambda+\mu}{n} \right\rfloor = \left\lfloor \frac{\sqrt{2}+1}{n} \right\rfloor$.

- (3) Note that $(2\mu, 2, \lambda)$ is a primitive positive solution of $2\alpha^2 + n\beta^2 = 4\gamma^2$. The result follows from arguments similar to (2).
- (4) In this case, $\mathbf{B}_{-2n} = \mathbf{A}_n$ and $\mathbf{B}_{-2n} \mathbf{1} = \mathbf{b}_{n,-1}$. Note that

$$m = -2n = 2(\lambda + 2\mu)^2 - (2\lambda + 2\mu)^2.$$

By Eq. (2.16), $r_4(K_2\mathcal{O}_{-2n}) = 0$ if and only if $\text{corank } \mathbf{A}_n = 1$ and $\mathbf{b}_{n,\lambda+2\mu} \notin \text{Im } \mathbf{A}_n$, if and only if $h_4(-n) = 1$ and

$$1 = \mathbf{1}^T \mathbf{b}_{n,\lambda+2\mu} = \left\lfloor \frac{\lambda + 2\mu}{n} \right\rfloor = \left\lfloor \frac{\sqrt{2} + 2}{n} \right\rfloor = \left\lfloor \frac{\sqrt{2} + 1}{n} \right\rfloor + \left\lfloor \frac{\sqrt{2}}{n} \right\rfloor,$$

i.e., $h_8(-n) + h_8(-2n) = 1$.

If $h_4(-n) = 1$, then $\text{corank } \mathbf{A}_n = 1$. Thus $\mathbf{A}_n \mathbf{x} = \mathbf{0}$ has two solutions, $\mathbf{A}_n \mathbf{x} = \mathbf{b}_{n, \lambda+2\mu}$ has at most two solutions. Thus implies that $r_4(K_2 \mathcal{O}_{-2n}) \leq 1$ by [Eq. \(2.16\)](#).

(5) The proof is similar to (4). $\square$

Proposition 2.9. *Let $n = p_1 \cdots p_k \equiv 1 \pmod{8}$ be a square-free positive integer with odd prime factors p_i such that $p_i \equiv \pm 1 \pmod{8}$ for all i . If $h_4(-n) = 1$, then*

$$h_8(-n) + h_8(-2n) \equiv \frac{d-1}{8} \pmod{2},$$

where d is the unique divisor of n such that $(d, n)_v = 1, \forall v$ and $d \neq 1, d \equiv 1 \pmod{4}$.

Proof. Notice that $d = \left(\frac{-1}{|d|}\right)|d|$ and

$$\begin{aligned} 0 &= [d, n]_{p_i} = [d, -1]_{p_i} + [d, -n]_{p_i} \\ &= [d, -1]_{p_i} + [|d|, -n]_{p_i} + \left[\frac{-1}{|d|}\right] [-1, -n]_{p_i} \\ &= [d, -1]_{p_i} + [|d|, -n]_{p_i} + \left[\frac{-1}{|d|}\right] \left[\frac{-1}{p_i}\right], \end{aligned}$$

we have

$$\begin{aligned} \mathbf{0} &= \mathbf{D}_{n,-1} \psi_n(|d|) + \mathbf{A}_n \psi_n(|d|) + \left[\frac{-1}{|d|}\right] \mathbf{b}_{n,-1} \\ &= (\mathbf{A}_n + \mathbf{D}_{n,-1}) \psi_n(|d|) + \mathbf{b}_{n,-1} \mathbf{b}_{n,-1}^T \psi_n(|d|) = \mathbf{A}_n^T \psi_n(|d|) \end{aligned}$$

by [Eq. \(2.17\)](#). Write $n = \lambda^2 - 2\mu^2$ where $\lambda, \mu \in \mathbb{N}$. By [Theorem 2.8\(2\)](#) and (3), $h_8(-n) + h_8(-2n) = 1$ if and only if

$$1 = \left[\frac{\lambda(\lambda + \mu)}{|d|}\right] = \left[\frac{1 + \mu/\lambda}{|d|}\right] = \left[\frac{2 + \sqrt{2}}{|d|}\right],$$

which is equivalent to $d \equiv 9 \pmod{16}$ by [\[Zha23, Lemma 5.4\]](#). $\square$

3. THE SELMER GROUPS AND THE CASSELS PAIRINGS

Let $n = PQ$ be a square-free positive integer with an ordered prime decomposition

$$n = \gcd(2, n) p_1 \cdots p_k q_1 \cdots q_\ell,$$

where $P = p_1 \cdots p_k, Q = q_1 \cdots q_\ell$. Assume that $p_1 \equiv \cdots \equiv p_k \equiv 1 \pmod{8}$ and there exists

$$\mathbf{u} = (u_1, \dots, u_k)^T \in \mathbb{F}_2^k, \quad \mathbf{v} = (v_1, \dots, v_\ell)^T \in \mathbb{F}_2^\ell$$

such that the Legendre symbol $\left[\frac{p_i}{q_j}\right] = u_i v_j$. Clearly,

$$\mathbf{1}^T \mathbf{u} = \sum_{i=1}^k u_i \quad \text{and} \quad \mathbf{1}^T \mathbf{v} = \sum_{j=1}^\ell v_j.$$

Lemma 3.1. *Assume that $\mathbf{1}^T \mathbf{u} = 0, \mathbf{1}^T \mathbf{v} = 1, p_1 \equiv \cdots \equiv p_k \equiv 1 \pmod{8}$ and Q is non-congruent with $\text{III}(E_Q)[2^\infty] = 0$. Then*

$$\text{Ker } \mathbf{M}_n = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \\ \mathbf{z} \\ \mathbf{0} \end{pmatrix} \mid \mathbf{x}, \mathbf{z} \in \text{Ker}(\mathbf{A}_P + \mathbf{U}_P) \right\}$$

In particular, $s_2(n) = 2 \operatorname{corank}(\mathbf{A}_P + \mathbf{U}_P)$.

Proof. Note that $\mathbf{A}_n \mathbf{1} = \mathbf{0}$ and $\mathbf{A}_P^T = \mathbf{A}_P$. By our assumptions,

$$\mathbf{A}_n = \begin{pmatrix} \mathbf{A}_P + \mathbf{U}_P & \mathbf{u}\mathbf{v}^T \\ \mathbf{v}\mathbf{u}^T & \mathbf{A}_Q \end{pmatrix} \quad \text{and} \quad \mathbf{A}_n^T = \begin{pmatrix} \mathbf{A}_P + \mathbf{U}_P & \mathbf{u}\mathbf{v}^T \\ \mathbf{v}\mathbf{u}^T & \mathbf{A}_Q^T \end{pmatrix}.$$

Note that $\mathbf{D}_{P,\pm 2} = \mathbf{O}_k$. If Q is odd, we have

$$\mathbf{M}_n = \begin{pmatrix} \mathbf{A}_P + \mathbf{U}_P & \mathbf{u}\mathbf{v}^T & \mathbf{O}_k & \mathbf{D}_{Q,2} \\ \mathbf{v}\mathbf{u}^T & \mathbf{A}_Q + \mathbf{D}_{Q,2} & \mathbf{A}_P + \mathbf{U}_P & \mathbf{u}\mathbf{v}^T \\ \mathbf{O}_k & \mathbf{D}_{Q,2} & \mathbf{v}\mathbf{u}^T & \mathbf{A}_Q + \mathbf{D}_{Q,-2} \end{pmatrix}.$$

If Q is even, we have

$$\mathbf{M}_n = \begin{pmatrix} \mathbf{A}_P + \mathbf{U}_P & \mathbf{u}\mathbf{v}^T & \mathbf{O}_k & \mathbf{D}_{Q,-1} \\ \mathbf{v}\mathbf{u}^T & \mathbf{A}_Q^T + \mathbf{D}_{Q,2} & \mathbf{A}_P + \mathbf{U}_P & \mathbf{u}\mathbf{v}^T \\ \mathbf{O}_k & \mathbf{D}_{Q,2} & \mathbf{v}\mathbf{u}^T & \mathbf{A}_Q + \mathbf{D}_{Q,2} \end{pmatrix}.$$

If

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \\ \mathbf{w} \end{pmatrix} \in \operatorname{Ker} \mathbf{M}_n,$$

then

$$(\mathbf{A}_P + \mathbf{U}_P)\mathbf{x} = \mathbf{u}\mathbf{v}^T\mathbf{y}, \quad (\mathbf{A}_P + \mathbf{U}_P)\mathbf{z} = \mathbf{u}\mathbf{v}^T\mathbf{w}$$

and

$$\mathbf{M}_Q \begin{pmatrix} \mathbf{y} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{v}\mathbf{u}^T\mathbf{x} \\ \mathbf{v}\mathbf{u}^T\mathbf{z} \end{pmatrix}.$$

Since $\mathbf{A}_P = \mathbf{A}_P^T$, we have $\mathbf{1}^T \mathbf{A}_P = \mathbf{0}^T$ and

$$0 = \mathbf{1}^T \mathbf{u}\mathbf{v}^T \mathbf{y} = \mathbf{1}^T (\mathbf{A}_P + \mathbf{U}_P) \mathbf{x} = \mathbf{1}^T \mathbf{U}_P \mathbf{x} = \mathbf{u}^T \mathbf{x}. \quad (3.1)$$

Similarly, $\mathbf{u}^T \mathbf{z} = 0$. Thus

$$\mathbf{M}_Q \begin{pmatrix} \mathbf{y} \\ \mathbf{w} \end{pmatrix} = \mathbf{0}.$$

Since $s_2(Q) = 0$, $\mathbf{M}_Q$ is invertible and we have $\mathbf{y} = \mathbf{w} = \mathbf{0}$. Thus $\mathbf{x}, \mathbf{z} \in \operatorname{Ker}(\mathbf{A}_P + \mathbf{U}_P)$,

$$\operatorname{Ker} \mathbf{M}_n = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \\ \mathbf{z} \\ \mathbf{0} \end{pmatrix} \mid \mathbf{x}, \mathbf{z} \in \operatorname{Ker}(\mathbf{A}_P + \mathbf{U}_P) \right\}$$

and $s_2(n) = 2 \operatorname{corank}(\mathbf{A}_P + \mathbf{U}_P)$. $\square$

Proposition 3.2. *Let f_i, f_j be two positive divisors of P such that $\gcd(f_i, f_j) = 1$ and $\psi_P(f_i), \psi_P(f_j) \in \operatorname{Ker}(\mathbf{A}_P + \mathbf{U}_P)$. Denote by*

$$\Lambda_t = (f_t, 1, f_t) \quad \text{and} \quad \Lambda'_t = (f_t, f_t, 1)$$

for $t = i, j$. Then

$$\begin{aligned}\langle \Lambda'_i, \Lambda_i \rangle &= \left\lfloor \frac{\sqrt{2}+1}{f_i} \right\rfloor + \left\lfloor \frac{\gamma_i}{f_i} \right\rfloor = \left\lfloor \frac{\sqrt{2}+1}{f_i} \right\rfloor + \left\lfloor \frac{\gamma'_i}{f_i} \right\rfloor, \\ \langle \Lambda'_i, \Lambda_j \rangle &= \left\lfloor \frac{\gamma_i}{f_j} \right\rfloor = \left\lfloor \frac{\gamma'_j}{f_i} \right\rfloor, \\ \langle \Lambda'_i, \Lambda'_i \rangle &= \left\lfloor \frac{\gamma_i \gamma'_i}{f_i} \right\rfloor, \quad \langle \Lambda'_i, \Lambda'_j \rangle = \left\lfloor \frac{\gamma_i \gamma'_j}{f_j} \right\rfloor,\end{aligned}$$

where $(\alpha_i, \beta_i, \gamma_i)$ (resp. $(\alpha'_i, \beta'_i, \gamma'_i)$) is a primitive positive solution of

$$f_i \alpha_i^2 + \frac{n}{f_i} \beta_i^2 = 4\gamma_i^2 \quad \left(\text{resp. } f_i \alpha_i'^2 - \frac{n}{f_i} \beta_i'^2 = 4\gamma_i'^2 \right).$$

Proof. Let $(\alpha''_i, \beta''_i, \gamma''_i)$ be a primitive positive solution of $f_i \alpha_i''^2 - \frac{2n}{f_i} \beta_i''^2 = 4\gamma_i''^2$. It's easy to see that $\alpha_i, \beta_i, \gamma_i, \alpha'_i, \beta'_i, \gamma'_i, \alpha''_i, \beta''_i, \gamma''_i$ are coprime to $n/\gcd(2, n)$.

(1) Recall that D_{Λ_i} is defined by

$$\begin{cases} H_1 : & -nt^2 + u_2^2 - f_i u_3^2 = 0, \\ H_2 : & -\frac{n}{f_i} t^2 + u_3^2 - u_1^2 = 0, \\ H_3 : & 2nt^2 + f_i u_1^2 - u_2^2 = 0. \end{cases}$$

Choose

$$\begin{aligned}Q_1 &= (\beta'_i, f_i \alpha'_i, 2\gamma'_i) \in H_1(\mathbb{Q}), & L_1 &= \frac{n}{f_i} \beta'_i t - \alpha'_i u_2 + 2\gamma'_i u_3, \\ Q_2 &= (0, 1, -1) \in H_2(\mathbb{Q}), & L_2 &= u_3 + u_1, \\ Q_3 &= (\beta''_i, 2\gamma''_i, f_i \alpha''_i) \in H_3(\mathbb{Q}), & L_3 &= \frac{2n}{f_i} \beta''_i t + 2\gamma''_i u_1 - \alpha''_i u_2.\end{aligned}$$

By Eq. (3.1), we have $\mathbf{u}^T \psi_P(f_t) = 0$, which implies that

$$\left[\frac{f_t}{q_s} \right] = \sum_{pr|f_t} u_r v_s = v_s \mathbf{u}^T \psi_P(f_t) = 0. \quad (3.2)$$

If $v = p_s \mid P$, then $\left[\frac{q_t}{p_s} \right] = \left[\frac{p_s}{q_t} \right] = u_s v_t$ and $p_s \equiv 1 \pmod{8}$. Thus we have

$$\left[\frac{Q}{p_s} \right] = u_s \mathbf{v}^T \mathbf{1} = u_s.$$

One can see that the s -th entry of the vector $(\mathbf{A}_P + \mathbf{U}_P) \psi_P(f_i)$ is

$$0 = u_s + \sum_{p|f_i} [p, -P]_{p_s} = \left[\frac{Q}{p_s} \right] + [f_i, -P]_{p_s} = \left[\frac{Q}{p_s} \right] + \left[\frac{P/f_i}{p_s} \right] = \left[\frac{n/f_i}{p_s} \right]$$

if $p_s \mid f_i$;

$$0 = \sum_{p|f_i} [p, -P]_{p_s} = [f_i, -P]_{p_s} = \left[\frac{f_i}{p_s} \right]. \quad (3.3)$$

if $p_s \mid \frac{P}{f_i}$.

(i) The case $v = p_s \mid f_i$. Take

$$P_v = (t, u_1, u_2, u_3) = (1, \sqrt{-2n/f_i}, 0, \sqrt{-n/f_i}).$$

Note that

$$(\beta'_i \sqrt{-n/f_i} + 2\gamma'_i)(-\beta'_i \sqrt{-n/f_i} + 2\gamma'_i) = f_i \alpha_i'^2$$

and one of $\pm \beta'_i \sqrt{-n/f_i} + 2\gamma'_i$ is congruent to $4\gamma'_i$ modulo v . Since $[f_i, f_t]_v = 0$ for $t = i, j$ by [Eq. \(3.3\)](#), we have

$$[\pm \beta'_i \sqrt{-n/f_i} + 2\gamma'_i, f_t]_v = [4\gamma'_i, f_t]_v.$$

Then

$$[L_1(P_v), f_t]_v = [4\gamma'_i \sqrt{-n/f_i}, f_t]_v = [\gamma'_i \sqrt{-n/f_i}, f_t]_v.$$

Similarly,

$$\begin{aligned} [L_2(P_v), f_t]_v &= [(\sqrt{2} + 1) \sqrt{-n/f_i}, f_t]_v, \\ [L_3(P_v), f_t]_v &= [4\sqrt{2}\gamma''_i \sqrt{-n/f_i}, f_t]_v = [\sqrt{2}\gamma''_i \sqrt{-n/f_i}, f_t]_v. \end{aligned}$$

Thus

$$\begin{aligned} [L_1 L_2(P_v), f_t]_v &= [(\sqrt{2} + 1)\gamma'_i, f_t]_v, \\ [L_1 L_3(P_v), f_t]_v &= [\sqrt{2}\gamma'_i \gamma''_i, f_t]_v. \end{aligned}$$

(ii) The case $v = p_s \mid \frac{P}{f_i}$. Take

$$P_v = (t, u_1, u_2, u_3) = (0, 1, \sqrt{f_i}, 1).$$

Similarly to [Section 3](#), we have

$$\begin{aligned} [L_1(P_v), f_t]_v &= [4\gamma'_i, f_t]_v = [\gamma'_i, f_t]_v, \\ [L_2(P_v), f_t]_v &= [2, f_t]_v = 0, \\ [L_3(P_v), f_t]_v &= [4\gamma''_i, f_t]_v = [\gamma''_i, f_t]_v, \end{aligned}$$

and then

$$\begin{aligned} [L_1 L_2(P_v), f_t]_v &= [\gamma'_i, f_t]_v, \\ [L_1 L_3(P_v), f_t]_v &= [\gamma'_i \gamma''_i, f_t]_v. \end{aligned}$$

By [Lemma 2.1](#) and [Eq. \(3.2\)](#), we have

$$\begin{aligned} \langle \Lambda_i, \Lambda_i \rangle &= \sum_{v|f_i} [\sqrt{2}\gamma'_i \gamma''_i, f_i]_v + \sum_{v|\frac{P}{f_i}} [\gamma'_i \gamma''_i, f_i]_v = \left[\frac{\sqrt{2}\gamma'_i \gamma''_i}{f_i} \right], \\ \langle \Lambda_i, \Lambda_j \rangle &= \sum_{v|f_i} [\sqrt{2}\gamma'_i \gamma''_i, f_j]_v + \sum_{v|\frac{P}{f_i}} [\gamma'_i \gamma''_i, f_j]_v = \left[\frac{\gamma'_i \gamma''_i}{f_j} \right], \\ \langle \Lambda_i, \Lambda'_i \rangle &= \sum_{v|f_i} [(\sqrt{2} + 1)\gamma'_i, f_i]_v + \sum_{v|\frac{P}{f_i}} [\gamma'_i, f_i]_v = \left[\frac{(\sqrt{2} + 1)\gamma'_i}{f_i} \right], \\ \langle \Lambda_i, \Lambda'_j \rangle &= \sum_{v|f_i} [(\sqrt{2} + 1)\gamma'_i, f_j]_v + \sum_{v|\frac{P}{f_i}} [\gamma'_i, f_j]_v = \left[\frac{\gamma'_i}{f_j} \right], \end{aligned} \tag{3.4}$$

(2) Recall that $D_{\Lambda'_i}$ is defined by

$$\begin{cases} H_1 : & -nt^2 + f_i u_2^2 - u_3^2 = 0, \\ H_2 : & -nt^2 + u_3^2 - f_i u_1^2 = 0, \\ H_3 : & \frac{2n}{f_i} t^2 + u_1^2 - u_2^2 = 0. \end{cases}$$

Choose

$$\begin{aligned} Q_1 &= (\beta_i, 2\gamma_i, f_i \alpha_i) \in H_1(\mathbb{Q}), & L_1 &= \frac{n}{f_i} \beta_i t - 2\gamma_i u_2 + \alpha_i u_3, \\ Q_2 &= (\beta'_i, f_i \alpha'_i, 2\gamma'_i) \in H_2(\mathbb{Q}), & L_2 &= \frac{n}{f_i} \beta'_i t - \alpha'_i u_3 + 2\gamma'_i u_1, \\ Q_3 &= (0, 1, -1) \in H_3(\mathbb{Q}), & L_3 &= u_1 + u_2. \end{aligned}$$

(i) The case $v \mid f_i$. Take

$$P_v = (t, u_1, u_2, u_3) = (1, \sqrt{-n/f_i}, \sqrt{n/f_i}, 0).$$

Similarly, we have

$$\begin{aligned} [L_1(P_v), f_t]_v &= [4\gamma_i \sqrt{n/f_i}, f_t]_v = [\gamma_i \sqrt{n/f_i}, f_t]_v, \\ [L_2(P_v), f_t]_v &= [4\gamma'_i \sqrt{-n/f_i}, f_t]_v = [\gamma'_i \sqrt{-n/f_i}, f_t]_v, \\ [L_3(P_v), f_t]_v &= [(\sqrt{-1} + 1) \sqrt{n/f_i}, f_t]_v, \end{aligned}$$

and then

$$\begin{aligned} [L_1 L_2(P_v), f_t]_v &= [\sqrt{-1} \gamma_i \gamma'_i, f_t]_v = [\gamma_i \gamma'_i, f_t]_v, \\ [L_1 L_3(P_v), f_t]_v &= [(\sqrt{-1} + 1) \gamma_i, f_t]_v = [(\sqrt{2} + 1) \gamma_i, f_t]_v. \end{aligned}$$

Here, we use the fact that

$$\begin{aligned} 4\sqrt{-1} &= (\sqrt{2} + \sqrt{-2})^2, \\ (\sqrt{2} + 1)(\sqrt{-1} + 1) &= \frac{1}{2}(\sqrt{2} + \sqrt{-1} + 1)^2 \end{aligned}$$

are squares in $\mathbb{Q}_v$.

(ii) The case $v \mid \frac{P}{f_i}$. Take

$$P_v = (t, u_1, u_2, u_3) = (0, 1, 1, \sqrt{f_i}).$$

Similarly, we have

$$\begin{aligned} [L_1(P_v), f_t]_v &= [-4\gamma_i, f_t]_v = [\gamma_i, f_t]_v, \\ [L_2(P_v), f_t]_v &= [4\gamma'_i, f_t]_v = [\gamma'_i, f_t]_v, \\ [L_3(P_v), f_t]_v &= [2, f_t]_v = 0, \end{aligned}$$

and then

$$\begin{aligned} [L_1 L_2(P_v), f_t]_v &= [\gamma_i \gamma'_i, f_t]_v, \\ [L_1 L_3(P_v), f_t]_v &= [\gamma_i, f_t]_v. \end{aligned}$$

By Lemma 2.1 and Eq. (3.2), we have

$$\begin{aligned}
\langle \Lambda'_i, \Lambda'_i \rangle &= \sum_{v|f_i} [\gamma_i \gamma'_i, f_i]_v + \sum_{v|\frac{P}{f_i}} [\gamma_i \gamma'_i, f_i]_v = \left[\frac{\gamma_i \gamma'_i}{f_i} \right], \\
\langle \Lambda'_i, \Lambda'_j \rangle &= \sum_{v|f_i} [\gamma_i \gamma'_i, f_j]_v + \sum_{v|\frac{P}{f_i}} [\gamma_i \gamma'_i, f_j]_v = \left[\frac{\gamma_i \gamma'_i}{f_j} \right], \\
\langle \Lambda'_i, \Lambda_i \rangle &= \sum_{v|f_i} [(\sqrt{2}+1)\gamma_i, f_i]_v + \sum_{v|\frac{P}{f_i}} [\gamma_i, f_i]_v = \left[\frac{(\sqrt{2}+1)\gamma_i}{f_i} \right], \\
\langle \Lambda'_i, \Lambda_j \rangle &= \sum_{v|f_i} [(\sqrt{2}+1)\gamma_i, f_j]_v + \sum_{v|\frac{P}{f_i}} [\gamma_i, f_j]_v = \left[\frac{\gamma_i}{f_j} \right],
\end{aligned} \tag{3.5}$$

Finally, we conclude the results by Eqs. (3.4) and (3.5). $\square$

4. PROOF OF MAIN THEOREMS

Lemma 4.1. *The following are equivalent:*

- n is non-congruent with $\text{III}(E_n)[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^{s_2(n)}$;
- the Cassels pairing on $\text{Sel}'_2(E_n)$ is non-degenerate.

Proof. The proof is due to [Wan16, pp 2146, 2157]. Since

$$0 \rightarrow E_n[2] \rightarrow E_n[4] \xrightarrow{\times 2} E_n[2] \rightarrow 0$$

is exact, we have the long exact sequence

$$0 \rightarrow \frac{E_n(\mathbb{Q})[2]}{2E_n(\mathbb{Q})[4]} \rightarrow \text{Sel}_2(E_n) \rightarrow \text{Sel}_4(E_n) \rightarrow \text{Im Sel}_4(E_n) \rightarrow 0,$$

where $\text{Im Sel}_4(E_n)$ is the image of $\text{Sel}_4(E_n) \xrightarrow{\times 2} \text{Sel}_2(E_n)$. It's known that the kernel of the Cassels pairing on $\text{Sel}_2(E_n)$ is $\text{Im Sel}_4(E_n)$. Thus

$$\text{rank}_{\mathbb{Z}} E_n(\mathbb{Q}) = 0, \quad \text{III}(E_n)[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^{s_2(n)}$$

if and only if $\#\text{Sel}_2(E_n) = \#\text{Sel}_4(E_n)$, if and only if $\text{Im Sel}_4(E_n) = E_n[2]$ in $\text{Sel}_2(E_n)$, if and only if the Cassels pairing on $\text{Sel}'_2(E_n)$ is non-degenerate. $\square$

Proof of Theorem 1.4. It follows from Lemma 3.1 that $s_2(n) = 0$ if and only if $\mathbf{A}_P + \mathbf{U}_P$ is invertible. This concludes the result. $\square$

Proof of Theorem 1.5. By Lemma 3.1, $s_2(n) = 2$ if and only if $\text{corank}(\mathbf{A}_P + \mathbf{U}_P) = 1$. Assume that $\text{corank}(\mathbf{A}_P + \mathbf{U}_P) = 1$ from now on. By our assumptions, $\psi_P(d)$ is a non-zero vector lying in $\text{Ker}(\mathbf{A}_P + \mathbf{U}_P)$. Then

$$\text{Ker } \mathbf{M}_n = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \psi_P(d) \\ 0 \end{pmatrix}, \begin{pmatrix} \psi_P(d) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \psi_P(d) \\ 0 \\ \psi_P(d) \\ 0 \end{pmatrix} \right\}.$$

Thus

$$\text{Sel}'_2(E_n) = \{(1, 1, 1), (d, 1, d), (1, d, d), (d, d, 1)\}$$

by Eqs. (2.5) and (2.7).

Denote by $\Lambda = (d, 1, d)$ and $\Lambda' = (d, d, 1)$. Then

$$\langle \Lambda, \Lambda' \rangle = \left\lfloor \frac{\sqrt{2}+1}{d} \right\rfloor + \left\lfloor \frac{\gamma}{d} \right\rfloor$$

by [Proposition 3.2](#). Hence the Cassels pairing on $\text{Sel}'_2(E_n)$ is non-degenerate if and only if $\left(\frac{\sqrt{2}+1}{d}\right)\left(\frac{\gamma}{d}\right) = -1$. Conclude the results by [Lemma 4.1](#). $\square$

Proof of [Corollary 1.6](#). Take $\mathbf{u} = \mathbf{0}$ and $\mathbf{v} = (1, 0, \dots, 0)^T$ in [Theorem 1.5](#), we obtain that $\mathbf{U}_P = \mathbf{0}$. Thus $\text{corank}(\mathbf{A}_P + \mathbf{U}_P) = 1$ if and only if $\text{corank } \mathbf{A}_P = 1$, if and only if $h_4(-P) = 1$ by [Eq. \(2.11\)](#).

Since $\mathbf{A}_P \mathbf{1} = \mathbf{0}$, the non-zero vector in $\text{Ker } \mathbf{A}_P$ is $\psi_P(d) = \mathbf{1}$. Thus $d = P$ and we conclude the result by [Theorem 2.8\(2\)](#) and [\(5\)](#). $\square$

Example 4.2. *We give two examples to show that our results produce new non-congruent numbers.*

- (1) Clearly, $\mathbf{M}_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Thus $q = 3$ is a non-congruent number with $\text{III}(E_3)[2^\infty] = 0$. If $p = 193$, then $\left(\frac{p}{q}\right) = 1$, $\mathbf{A}_p = 0$ and $h_4(-p) = 1$. Since $52^2 \equiv 2 \pmod{p}$, we have

$$h_8(-p) = 1 - \left\lfloor \frac{\sqrt{2}+1}{p} \right\rfloor = 1 - \left\lfloor \frac{53}{193} \right\rfloor = 0.$$

Since $193 \times 1^2 + 3 \times 1^2 = 4 \times 7^2$ and $\left(\frac{7}{p}\right) = 1$, we obtain that $n = pq = 3 \times 193$ is non-congruent with $\text{III}(E_n)[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2$ by [Corollary 1.6](#).

- (2) Clearly, $\mathbf{M}_{10} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Thus $Q = 2q = 10$ is a non-congruent number with $\text{III}(E_{10})[2^\infty] = 0$. If $p = 241 = 23^2 - 2 \times 12^2$, then $\left(\frac{p}{q}\right) = 1$, $\mathbf{A}_p = 0$ and $h_4(-p) = 1$. Since $22^2 \equiv 2 \pmod{p}$, we have

$$h_8(-p) = 1 - \left\lfloor \frac{\sqrt{2}+1}{p} \right\rfloor = 1 - \left\lfloor \frac{23}{241} \right\rfloor = 0.$$

Since $241 \times 2^2 + 10 \times 24^2 = 4 \times 41^2$ and $\left(\frac{41}{p}\right) = 1$, we obtain that $n = 2pq = 10 \times 241$ is non-congruent with $\text{III}(E_n)[2^\infty] \cong (\mathbb{Z}/2\mathbb{Z})^2$ by [Corollary 1.6](#).

Proof of [Corollary 1.7](#). (1) Note that $(\alpha, \beta, \gamma) = (4, 2n-2, n+1)$ is a positive primitive solution of $n\alpha^2 + \beta^2 = 4\gamma^2$. Thus $\left[\frac{\gamma}{n}\right] = \left[\frac{n+1}{n}\right] = 0$. This concludes the result by [Corollary 1.6](#) and [Theorem 2.8\(5\)](#).

- (2) Write $n = \lambda^2 - 2\mu^2$ where $\lambda, \mu \in \mathbb{N}$. Then $(2, 2\mu, \lambda)$ is a primitive positive solution of $n\alpha^2 + 2\beta^2 = 4\gamma^2$. By [Theorem 2.8\(3\)](#), $\left[\frac{\lambda}{n}\right] = 1 - h_8(-2n)$. This concludes the result by [Theorem 2.8\(4\)](#) and [Corollary 1.6](#). $\square$

Proof of [Theorem 1.8](#). By our assumptions (we rearrange the order of prime factors of P),

$$\mathbf{A}_P + \mathbf{U}_P = \mathbf{A}_P = \text{diag}\{\mathbf{A}_{f_1}, \dots, \mathbf{A}_{f_r}\}.$$

Since $h_4(-f_i) = 1$, we have $\text{corank } \mathbf{A}_{f_i} = 1$ by [Theorem 2.8\(1\)](#). Since $\mathbf{A}_{f_i} \mathbf{1} = \mathbf{0}$, we have $s_2(n) = 2r$ and the kernel of $\mathbf{M}_n$ consists of vectors

$$\begin{pmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{c}_r \\ \mathbf{0} \\ \mathbf{d}_1 \\ \vdots \\ \mathbf{d}_r \\ \mathbf{0} \end{pmatrix},$$

where $\mathbf{c}_i, \mathbf{d}_i = \mathbf{0}$ or $\mathbf{1}$ are vectors in $\text{Ker } \mathbf{A}_{f_i}$. Thus $\text{Sel}'_2(E_n)$ is generated by $\Lambda_1, \dots, \Lambda_r, \Lambda'_1, \dots, \Lambda'_r$, where

$$\Lambda_i = (f_i, 1, f_i), \quad \Lambda'_i = (f_i, f_i, 1)$$

by [Eqs. \(2.5\)](#) and [\(2.7\)](#). By [Proposition 3.2](#), we have $\left[\frac{\gamma'_i}{f_j}\right] = \left[\frac{\gamma_j}{f_i}\right]$ and the Cassels pairing with respect to this basis is

$$\mathbf{X} = \begin{pmatrix} * & \mathbf{B}^T + \mathbf{C} \\ \mathbf{B} + \mathbf{C} & \mathbf{B} + \mathbf{B}^T \end{pmatrix},$$

where

$$\mathbf{B} = \left(\left[\frac{\gamma_i}{f_j} \right] \right)_{r \times r} \quad \text{and} \quad \mathbf{C} = \text{diag} \left\{ \left[\frac{\sqrt{2}+1}{f_1} \right], \dots, \left[\frac{\sqrt{2}+1}{f_r} \right] \right\}.$$

Since $h_4(-f_i) = 1$, we have

$$\mathbf{C} = \text{diag} \{ 1 - h_8(-f_1), \dots, 1 - h_8(-f_r) \}$$

by [Theorem 2.8\(2\)](#). By our assumptions,

$$\mathbf{B} = \text{diag} \{ h_8(-f_1), \dots, h_8(-f_r) \}.$$

Therefore, $\mathbf{X} = \begin{pmatrix} * & \mathbf{I} \\ \mathbf{I} & \mathbf{O} \end{pmatrix}$ is invertible, i.e., the Cassels pairing on $\text{Sel}'_2(E_n)$ is non-degenerate. Conclude the results by [Lemma 4.1](#). $\square$

Proof of [Corollary 1.9](#). (1) Since

$$\mathbf{R}_{-n} = \text{diag} \{ \mathbf{A}_n, 0 \} = \text{diag} \{ \mathbf{A}_{f_1}, \dots, \mathbf{A}_{f_r}, 0 \},$$

we have $h_4(-n) = r$ and $\mathcal{A}_{-n}[2] \cap \mathcal{A}_{-n}^2$ is generated by $\theta_{-n}(f_1), \dots, \theta_{-n}(f_{r-1})$ and $\theta_{-n}(2)$ by [Eqs. \(2.10\)](#) and [\(2.11\)](#). Here, one notice that

$$\theta_{-n}(f_1) \cdots \theta_{-n}(f_r) = \theta_{-n}(n) = [(\sqrt{-n})]$$

is the trivial class. If $h_8(-n) = r$, or $h_8(-n) = r - 1$ and $[(2, \sqrt{-n})] \notin \mathcal{A}_{-n}^4$, then all $\theta_{-n}(f_i) \in \mathcal{A}_{-n}[2] \cap \mathcal{A}_{-n}^4$. By [Proposition 2.4](#), this implies that $\mathbf{b}_{n, \gamma_i} \in \text{Im } \mathbf{A}_n$, where $(\alpha_i, \beta_i, \gamma_i)$ is a primitive positive solution of $f_i \alpha_i^2 - \frac{n}{f_i} \beta_i^2 = 4\gamma_i^2$. Thus $\mathbf{b}_{f_j, \gamma_i} \in \text{Im } \mathbf{A}_{f_j}$ for all j . Since $\mathbf{1}^T \mathbf{A}_{f_j} = \mathbf{0}^T$, we have

$$0 = \mathbf{1}^T \mathbf{b}_{f_j, \gamma_i} = \left[\frac{\gamma_i}{f_j} \right].$$

Conclude the results by [Theorem 1.8](#).

- (2) Similar to (1), $h_4(-2n) = r$ and $\mathcal{A}_{-2n}[2] \cap \mathcal{A}_{-2n}^2$ is generated by $\theta_{-2n}(f_1), \dots, \theta_{-2n}(f_r)$ by Eqs. (2.10) and (2.11). Here, one notice that

$$\theta_{-2n}(2) = \theta_{-2n}(f_1) \cdots \theta_{-2n}(f_r)$$

since $\theta_{-2n}(2n) = [(\sqrt{-2n})]$ is the trivial class. If $h_8(-2n) = r$, then all $\theta_{-2n}(f_i) \in \mathcal{A}_{-2n}[2] \cap \mathcal{A}_{-2n}^4$. One can conclude the results similar to (1). $\square$

This paper reveals a new phenomenon: for a general non-congruent number n with the second minimal 2-primary Shafarevich group, the criterion cannot be expressed solely in terms of the 4-ranks and 8-ranks of class groups of quadratic fields, even though this is possible when the prime factors of n lie in certain residue classes. A key remaining problem is how to find simple arithmetic conditions that characterize non-congruent numbers with specific 2-primary Shafarevich groups.

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