

THE GENERATING FIELDS OF TWISTED KLOOSTERMAN SUMS

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ABSTRACT. We use the Kloosterman sheaves constructed by Fisher to show when two twisted Kloosterman sums differ by a factor of a $(q-1)$ -th root of unity, and use p -adic analysis to prove the non-vanishing of twisted Kloosterman sums. Then we determine generating fields of twisted Kloosterman sums by these results.

1. INTRODUCTION

1.1. Background. Let p be a prime number, $q = p^d$ a power of p , and $\mathbb{F}_q$ the field with q elements. Denote by $\mu_n \subseteq \overline{\mathbb{Q}}^\times$ the group of n -th roots of unity. Let $\psi : \mathbb{F}_p \rightarrow \mu_p$ be a fixed non-trivial additive character. For $\chi = \{\chi_1, \dots, \chi_n\}$ an unordered n -tuple of multiplicative characters $\chi_i : \mathbb{F}_q^\times \rightarrow \mu_{q-1}$ and $a \in \mathbb{F}_q^\times$, define the *Kloosterman sum* as

$$\text{Kl}_n(\psi, \chi, q, a) = \sum_{\substack{x_1 \cdots x_n = a \\ x_i \in \mathbb{F}_q^\times}} \chi_1(x_1) \cdots \chi_n(x_n) \psi(\text{Tr}(x_1 + \cdots + x_n)),$$

where $\text{Tr} = \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$. Clearly it lies in $\mathbb{Z}[\mu_{p(q-1)}]$.

When $\chi = \mathbf{1} = \{1, \dots, 1\}$ is trivial, the distinctness of Kloosterman sums is studied by many peoples. It's easy to see that

$$a, b \text{ conjugate} \implies \text{Kl}_n(\psi, \mathbf{1}, q, a) = \text{Kl}_n(\psi, \mathbf{1}, q, b).$$

Fisher in [Fis92, Remark 4.28(2)] conjectured that the converse

$$\text{Kl}_n(\psi, \mathbf{1}, q, a) = \text{Kl}_n(\psi, \mathbf{1}, q, b) \implies a, b \text{ conjugate} \tag{1.1}$$

is also true if $p \geq nd$. It's known that [Eq. \(1.1\)](#) holds when $p > (2n^{2d} + 1)^2$ in [Fis92], or $p \geq (d-1)n + 2$ and p does not divide a certain integer in [Wan95, Theorem 1.3]. Once [Eq. \(1.1\)](#) holds, one can obtain that $\text{Kl}_n(\psi, \mathbf{1}, q, a)$ generates $\mathbb{Q}(\mu_p)^H$, where

$$H = \left\{ t \in \text{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q}) \mid \exists k \in \mathbb{Z} \text{ such that } t^n = a^{1-p^k} \right\}.$$

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1.2. Notations and main results. In this article, we will study the *generating fields* of twisted Kloosterman sums. We need the following notations:

- $c = c(\chi) \mid (q-1)$ the minimal positive integer such that $\chi_i^c = 1, i = 1, \dots, n$, i.e., the least common multiplier of orders of χ_i .
- $\chi^w := \{\chi_1^w, \dots, \chi_n^w\}$, where $w \in \mathbb{Z}$ or $\mathbb{Z}/c\mathbb{Z}$.
- $\chi\eta := \{\chi_1\eta, \dots, \chi_n\eta\}$, where η is a multiplicative character.
- $\chi \circ \sigma := \{\chi_1 \circ \sigma, \dots, \chi_n \circ \sigma\}$, where $\sigma \in \text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$.
- $\prod \chi := \chi_1 \cdots \chi_n$.

Clearly, the Galois group

$$\text{Gal}(\mathbb{Q}(\mu_{pc})/\mathbb{Q}) = \left\{ \sigma_t \tau_w \mid t \in (\mathbb{Z}/p\mathbb{Z})^\times, w \in (\mathbb{Z}/c\mathbb{Z})^\times \right\},$$

where

$$\sigma_t(\zeta_p) = \zeta_p^t, \sigma_t(\zeta_c) = \zeta_c \quad \tau_w(\zeta_p) = \zeta_p, \tau_w(\zeta_c) = \zeta_c^w$$

for any $\zeta_p \in \mu_p, \zeta_c \in \mu_c$.

Theorem 1.1. *Assume that $p > \max\{(2n^{2d} + 1)^2, (3n - 1)c - n\}$ and for any i, j , $\chi_i = \chi_j$ if $\chi_i^n = \chi_j^n$. Then $\text{Kl}_n(\psi, \chi, q, a)$ generates $\mathbb{Q}(\mu_{pc})^H$, where H consists of those $\sigma_t \tau_w$ such that there exists an integer k and a character η satisfying*

$$t^n = a^{1-p^k}, \quad \chi^w = \chi^{p^k} \eta, \quad \eta(a) = \prod \chi^w(t).$$

A basic observation tells that

$$\sigma_t \tau_w \text{Kl}_n(\psi, \chi, q, a) = \prod \chi(t)^{-w} \text{Kl}_n(\psi, \chi^w, q, at^n).$$

To study the generating fields, we need to know when two twisted Kloosterman sums differ by a factor of some $\lambda \in \mu_{q-1}$. In [Section 2](#), we will recall the construction of Kloosterman sheaves by Fisher and show when two twisted Kloosterman sums differ by a factor of λ for sufficiently large p , see [Theorem 2.7](#). We also need the non-vanishing of twisted Kloosterman sums, which will be proved by p -adic analysis in [Section 3](#). Then we will finish the proof in [Section 4](#) and end this paper with several examples in [Section 5](#).

2. KLOOSTERMAN SHEAVES AND FISHER'S DESCENT

2.1. Kloosterman sheaves. Let $\ell \neq p$ be a prime and fix an embedding $\overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$. Then the additive and multiplicative characters ψ, χ_i can take value both in $\overline{\mathbb{Q}}_\ell$ or $\mathbb{C}$.

Deligne in [\[Del77, Theorem 7.8\]](#) and Katz in [\[Kat88, Theorem 4.11\]](#) defined the Kloosterman sheaf of $\overline{\mathbb{Q}}_\ell$ -modules

$$\mathcal{K}\ell = \mathcal{K}\ell_{n,q}(\psi, \chi)$$

on $\mathbb{G}_m/\mathbb{F}_q$, with the following properties:

- $\mathcal{K}\ell$ is lisse of rank n and pure of weight $n - 1$.
- For any $a \in \mathbb{F}_q^\times$, $\text{Tr}(\text{Frob}_a, \mathcal{K}\ell_{\overline{a}}) = (-1)^{n-1} \text{Kl}_n(\psi, \chi, q, a)$.
- $\mathcal{K}\ell$ is tame at 0.
- $\mathcal{K}\ell$ is totally wild with Swan conductor 1 at ∞ . So all ∞ -breaks are $1/n$.

Here Frob_a denotes the geometric Frobenius at a .

Definition 2.1. The n -tuple χ is called *Kummer-induced* if there exists a non-trivial character Λ such that $\chi = \chi\Lambda$ as unordered n -tuples.

Remark 2.2. If χ is Kummer-induced, then $\prod \chi = \prod(\chi\Lambda) = \Lambda^n \prod \chi$, $\Lambda^n = 1$. Since

$$\mathrm{Kl}_n(\psi, \chi\eta, q, a) = \eta(a) \mathrm{Kl}_n(\psi, \chi, q, a),$$

we have $\mathrm{Kl}_n(\psi, \chi, q, a) = 0$ if χ is Kummer-induced and $\Lambda(a) \neq 1$. See [Fis92, Remark 1.6].

Remark 2.3. When χ is not Kummer-induced, $\mathcal{K}\ell$ is not *geometrically Kummer-induced*. That's to say, $\mathcal{K}\ell \mid \mathbb{G}_m/\overline{\mathbb{F}}_p$ is not of type $(t \mapsto t^N)_*\mathcal{F}$ for some integer $N > 1$ and some lisse sheaf $\mathcal{F}$ on $\mathbb{G}_m/\overline{\mathbb{F}}_p$. See [Fis92, Theorem 2.9].

2.2. Fisher's descent. In [Fis92, Theorem 3.12], Fisher gave a descent of Kloosterman sheaves along an extension of finite fields. For any $a \in \mathbb{F}_q^\times$, he defined a lisse sheaf $\mathcal{F}_a(\chi)$ on $\mathbb{G}_m = \mathbb{G}_m/\mathbb{F}_p$, such that

$$\mathcal{F}_a(\chi) \mid \mathbb{G}_m/\mathbb{F}_q = \bigotimes_{\sigma \in \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)} (t \mapsto \sigma(a)t^n)^* \mathcal{K}\ell_{n,q}(\psi \circ \sigma^{-1}, \chi \circ \sigma^{-1}).$$

Moreover,

- $\mathcal{F}_a(\chi)$ is lisse of rank n^d and pure of weight $d(n-1)$.
- For any $t \in \mathbb{F}_p^\times$, $\mathrm{Tr}(\mathrm{Frob}_t, \mathcal{F}_a(\chi)_{\bar{t}}) = (-1)^{(n-1)d} \mathrm{Kl}_n(\psi, \chi, q, at^n)$.
- $\mathcal{F}_a(\chi)$ is tame at 0 and its ∞ -breaks are at most 1.

2.3. Distinctness. We will consider when

$$\mathrm{Kl}_n(\psi, \chi, q, a) = \lambda \mathrm{Kl}_n(\psi, \rho, q, b)$$

for some $\lambda \in \mu_{q-1}$. The argument almost follows [Fis92], while $\lambda = 1$ in his paper.

For a lisse sheaf $\mathcal{F}$ on $\mathbb{G}_m$, denote by $G_{\mathrm{geom}}(\mathcal{F})$ the geometric monodromy group of $\mathcal{F}$, i.e., the Zariski closure of $\pi_1(\mathbb{G}_m/\overline{\mathbb{F}}_p)$ in $\mathrm{GL}(\mathcal{F})$. Let $G_{\mathrm{geom}}(\mathcal{F})^\circ$ be the connected component of $G_{\mathrm{geom}}(\mathcal{F})$ and $\mathfrak{g}(\mathcal{F})$ its Lie algebra.

Proposition 2.4 ([Fis92, Proposition 4.18]). *Assume that $p > 2n+1$ and χ is not Kummer-induced.*

- (1) *As a representation of $\mathfrak{g}(\mathcal{F}_a(\chi))$, $\mathcal{F}_a(\chi)$ has a highest weight $\lambda_a(\chi)$ with multiplicity one.*
- (2) *$\mathcal{F}_a(\chi)$ has a geometrically irreducible sub-sheaf $\mathcal{G}_a(\chi)$, such that as a representation of $\mathfrak{g}(\mathcal{F}_a(\chi))$, $\mathcal{G}_a(\chi)$ is an irreducible sub-representation with unique highest weight $\lambda_a(\chi)$. Moreover, $\mathcal{G}_a(\chi) \mid \mathbb{G}_m/\overline{\mathbb{F}}_p$ occurs exactly once in $\mathcal{F}_a(\chi) \mid \mathbb{G}_m/\overline{\mathbb{F}}_p$.*

The multiplicative character χ can be viewed as a character on $\mathbb{F}_p$ -points of $\mathbb{B}^\times = \mathrm{Res}_{\mathbb{F}_q/\mathbb{F}_p} \mathbb{G}_m$. It gives a rank one lisse sheaf on $\mathbb{B}^\times$ constructed from the Lang torsor as in [Kat88, §4.3]. Denote by $\mathcal{L}_\psi$ its restriction on $\mathbb{G}_m$. Similarly, the additive character ψ gives a rank one lisse sheaf on $\mathbb{G}_a/\mathbb{F}_p$. Denote by $\mathcal{L}_\psi$ its restriction on $\mathbb{G}_m$. For any $t \in \mathbb{F}_p^\times$,

$$\mathrm{Tr}(\mathrm{Frob}_t, (\mathcal{L}_\chi)_{\bar{t}}) = \chi(t), \quad \mathrm{Tr}(\mathrm{Frob}_t, (\mathcal{L}_\psi)_{\bar{t}}) = \psi(t).$$

Lemma 2.5 ([Fis92, Lemma 4.9]). *Let $\mathcal{F}, \mathcal{F}'$ be lisse sheaves on $\mathbb{G}_m$ of same rank r and pure of the same weight w . Assume that for any $t \in \mathbb{F}_p^\times$,*

$$\mathrm{Tr}(\mathrm{Frob}_t, \mathcal{F}_t) = \mathrm{Tr}(\mathrm{Frob}_t, \mathcal{F}'_t).$$

Let $\mathcal{G}$ be a geometrically irreducible sheaf of rank s on $\mathbb{G}_m$, pure of weight w , such that $\mathcal{G} \mid \mathbb{G}_m/\overline{\mathbb{F}}_p$ occurs exactly once in $\mathcal{F} \mid \mathbb{G}_m/\overline{\mathbb{F}}_p$. Then $\mathcal{G} \mid \mathbb{G}_m/\overline{\mathbb{F}}_p$ occurs at least once in $\mathcal{F}' \mid \mathbb{G}_m/\overline{\mathbb{F}}_p$, provided that $p > (2rs(M_0 + M_\infty) + 1)^2$, where M_η is the largest η -break of $\mathcal{F} \oplus \mathcal{F}'$.

Proposition 2.6 ([Fis92, Corollary 4.20]). Let $a, b \in \mathbb{F}_q^\times$ and let χ, ρ be n -tuples of multiplicative characters $\chi_i, \rho_j : \mathbb{F}_q^\times \rightarrow \overline{\mathbb{Q}}_\ell^\times$ respectively. Assume that $p > (2n^{2d} + 1)^2$, χ is not Kummer-induced and there is $\lambda \in \mu_{q-1}$ such that

$$\text{Kl}_n(\psi, \chi, q, a) = \lambda \text{Kl}_n(\psi, \rho, q, b).$$

Then $\mathcal{G}_a(\chi) \otimes \mathcal{L}_{\Pi \bar{\chi}} \mid \mathbb{G}_m/\overline{\mathbb{F}}_p$ occurs at least once in $\mathcal{F}_b(\rho) \otimes \mathcal{L}_{\Pi \bar{\rho}} \mid \mathbb{G}_m/\overline{\mathbb{F}}_p$.

Proof. Denote by

$$\mathcal{F} = \mathcal{F}_a(\chi) \otimes \mathcal{L}_{\Pi \bar{\chi}}, \quad \mathcal{F}' = \mathcal{F}_b(\rho) \otimes \mathcal{L}_{\Pi \bar{\rho}}, \quad \mathcal{G} = \mathcal{G}_a(\chi) \otimes \mathcal{L}_{\Pi \bar{\chi}}.$$

For any $t \in \mathbb{F}_p^\times$, we have $\sigma_t \lambda = \lambda$. Since

$$\begin{aligned} \sigma_t(\text{Kl}_n(\psi, \chi, q, a)) &= \prod \bar{\chi}(t) \cdot \text{Kl}_n(\psi, \chi, q, at^n) = (-1)^{(n-1)d} \text{Tr}(\text{Frob}_t, \mathcal{F}_{\bar{t}}), \\ \sigma_t(\text{Kl}_n(\psi, \rho, q, b)) &= \prod \bar{\rho}(t) \cdot \text{Kl}_n(\psi, \rho, q, bt^n) = (-1)^{(n-1)d} \text{Tr}(\text{Frob}_t, \mathcal{F}'_{\bar{t}}), \end{aligned}$$

we have $\text{Tr}(\text{Frob}_t, \mathcal{F}_{\bar{t}}) = \lambda \text{Tr}(\text{Frob}_t, \mathcal{F}'_{\bar{t}})$.

Let $V = \overline{\mathbb{Q}}_\ell \cdot e$ with $\text{Frob}_p \cdot e = \lambda e$, where $\text{Frob}_p \in \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ denotes the geometric Frobenius. Denote by $\mathcal{L}_0$ the sheaf on $\text{Spec } \mathbb{F}_p$ corresponding to this module and let $\mathcal{L}$ be its pulling-back along $\mathbb{G}_m \rightarrow \text{Spec } \mathbb{F}_p$. Then for any $t \in \mathbb{F}_p^\times$,

$$\text{Tr}(\text{Frob}_t, \mathcal{L}_{\bar{t}}) = \text{Tr}(\text{Frob}_p, \mathcal{L}_0) = \lambda, \quad \text{Tr}(\text{Frob}_t, (\mathcal{F}' \otimes \mathcal{L})_{\bar{t}}) = \text{Tr}(\text{Frob}_t, \mathcal{F}'_{\bar{t}}).$$

Since $\mathcal{L} \mid \mathbb{G}_m/\overline{\mathbb{F}}_p$ is trivial, the result then follows by applying [Lemma 2.5](#) to sheaves $\mathcal{F}, \mathcal{F}' \otimes \mathcal{L}, \mathcal{G}$ with $r = s = n^d, M_0 = 0$ and $M_\infty \leq 1$. $\square$

Theorem 2.7. Let $a, b \in \mathbb{F}_q^\times$ and let χ, ρ be n -tuples of multiplicative characters. Assume that χ, ρ are not Kummer-induced and neither of them is of type $\{\xi_1, \xi_1^{-1}, 1, \Lambda_2\} \xi_2$. If $p > (2n^{2d} + 1)^2$ and

$$\text{Kl}_n(\psi, \chi, q, a) = \lambda \text{Kl}_n(\psi, \rho, q, b)$$

for some $\lambda \in \mu_{q-1}$, then there exists $\sigma \in \text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ and a multiplicative character η , such that $b = \sigma(a)$ and $\rho = (\chi \circ \sigma^{-1})\eta$ as unordered tuples. Moreover, either both Kloosterman sums vanish or $\eta(b) = \lambda^{-1}$.

Here, Λ_2 denotes the non-trivial quadratic character of $\mathbb{F}_q^\times$.

Proof. Denote by

$$\mathcal{H} = \mathcal{K}\ell_{n,q}(\psi, \chi) \mid \mathbb{G}_m/\overline{\mathbb{F}}_p \quad \text{and} \quad \mathcal{K} = \mathcal{K}\ell_{n,q}(\psi, \rho) \mid \mathbb{G}_m/\overline{\mathbb{F}}_p.$$

By our assumptions, $\mathcal{H}$ and $\mathcal{K}$ are not Kummer-induced by [\[Fis92, Theorem 2.9\]](#).

By applying [\[Kat90, Theorems 8.8.1, 8.11.3\]](#) with $n = 4, m = 0$, we obtain that $G_{\text{geom}}(\mathcal{H})^\circ = \text{SO}(4)$ if and only if there is a multiplicative character η such that $\chi = \bar{\chi}\eta = \chi^{-1}\eta$ as unordered 4-tuples and $\prod \chi = \Lambda_2\eta^2$. In which case, there is a permutation $\varepsilon \in S_4$ such that $\chi_i \chi_{\varepsilon(i)} = \eta$.

- If $\varepsilon = 1$, then $\chi_i^2 = \eta$, $\chi_i = \chi_1$ or $\chi_1 \Lambda_2$. Since $\prod \chi = \Lambda_2\eta^2$, we have $\chi = \{1, 1, 1, \Lambda_2\}\chi_1$ or $\{1, 1, 1, \Lambda_2\}\chi_1 \Lambda_2$.

- If $\varepsilon = (1234)$ or $(12)(34)$, then $\chi_1\chi_2 = \chi_3\chi_4 = \eta$, which contradicts to $\prod \chi = \Lambda_2\eta^2$.
- If $\varepsilon = (123)$, then $\chi_1\chi_2 = \chi_2\chi_3 = \chi_3\chi_1 = \eta$, $\chi_1 = \chi_2 = \chi_3$. Since $\prod \chi = \Lambda_2\eta^2 = \Lambda_2\chi_1^4$, we have $\chi_4 = \chi_1\Lambda_2$ and $\chi = \{1, 1, 1, \Lambda_2\}\chi_1$.
- If $\varepsilon = (12)$, then $\chi_1\chi_2 = \eta$, $\chi_3^2 = \chi_4^2 = \eta$. Since $\prod \chi = \Lambda_2\eta^2$, $\chi_3\chi_4 = \Lambda_2\eta = \Lambda_2\chi_3^2$, we have $\chi_4 = \Lambda_2\chi_3$. Therefore,

$$\chi = \{\chi_1, \chi_3^2\chi_1^{-1}, \chi_3, \chi_3\Lambda_2\} = \{\chi_1\chi_3^{-1}, \chi_1^{-1}\chi_3, 1, \Lambda_2\}\chi_3.$$

- The remaining cases can be discussed similarly.

Since the form of χ contradicts our assumptions, we have $G_{\text{geom}}(\mathcal{H})^\circ \neq \text{SO}(4)$. Similarly, $G_{\text{geom}}(\mathcal{K})^\circ \neq \text{SO}(4)$.

The following argument follows from [Fis92, Theorem 4.22]. For $a \in \overline{\mathbb{F}}_p^\times$, denote by $T_a : t \mapsto at$ a translation on $\mathbb{G}_{m/\overline{\mathbb{F}}_p}$ and

$$\mathcal{H}_\sigma := T_{\sigma(a)}^*(\mathcal{H} \circ \sigma^{-1}), \quad \mathcal{K}_\tau := T_{\tau(b)}^*(\mathcal{K} \circ \tau^{-1}).$$

Let G be the geometric monodromy group of

$$\bigoplus_{\sigma \in \text{Gal}(\overline{\mathbb{F}}_q/\overline{\mathbb{F}}_p)} \mathcal{H}_\sigma \oplus \bigoplus_{\tau \in \text{Gal}(\overline{\mathbb{F}}_q/\overline{\mathbb{F}}_p)} \mathcal{K}_\tau,$$

and $\mathfrak{g}$ the Lie algebra of G° . Since $G_{\text{geom}}(\mathcal{H}) \neq \text{SO}(4)$, we have $G_{\text{geom}}(\mathcal{H}_\sigma) \neq \text{SO}(4)$ for any σ . This implies that $\mathfrak{g}(\mathcal{H}_\sigma)$ is simple. Let λ_σ (resp. μ_τ) denote the highest weight of $\mathcal{H}_\sigma$ (resp. $\mathcal{K}_\tau$). Since

$$\mathcal{F}_a(\chi) \mid \mathbb{G}_{m/\overline{\mathbb{F}}_q} = \bigotimes_{\sigma \in \text{Gal}(\overline{\mathbb{F}}_q/\overline{\mathbb{F}}_p)} (t \mapsto t^n)^* \mathcal{H}_\sigma,$$

we have $\lambda_a(\chi) = \sum_\sigma \lambda_\sigma$, $\lambda_b(\rho) = \sum_\tau \mu_\tau$. By Proposition 2.6, we have

$$\begin{aligned} \mathcal{G}_a(\chi) \otimes \mathcal{L}_{\Pi \bar{\chi}} \mid \mathbb{G}_{m/\overline{\mathbb{F}}_p} &\hookrightarrow \mathcal{F}_b(\rho) \otimes \mathcal{L}_{\Pi \bar{\rho}} \mid \mathbb{G}_{m/\overline{\mathbb{F}}_p}, \\ \mathcal{G}_b(\rho) \otimes \mathcal{L}_{\Pi \bar{\rho}} \mid \mathbb{G}_{m/\overline{\mathbb{F}}_p} &\hookrightarrow \mathcal{F}_a(\chi) \otimes \mathcal{L}_{\Pi \bar{\chi}} \mid \mathbb{G}_{m/\overline{\mathbb{F}}_p}. \end{aligned}$$

Since as representations of $\mathfrak{g}$, $\mathcal{G}_a(\chi)$, $\mathcal{F}_a(\chi)$ have the highest weight $\lambda_a(\chi)$, $\mathcal{G}_b(\rho)$, $\mathcal{F}_b(\rho)$ have the highest weight $\lambda_b(\rho)$, we have $\lambda_a(\chi) = \lambda_b(\rho)$. Since λ_σ, μ_τ are fundamental weights, this implies that there is a σ such that $\lambda_\sigma = \mu_1$. Therefore, $\mathcal{H}_\sigma \cong \mathcal{K}_1$ as representations of $\mathfrak{g}$, and $\mathcal{H}_\sigma \otimes \mathcal{L} \cong \mathcal{K}_1$ as sheaves on $\mathbb{G}_{m/\overline{\mathbb{F}}_p}$ for some $\mathcal{L}$. By [Kat90, Lemma 8.11.7.1], $\mathcal{L} = \mathcal{L}_\eta$ for some tame character η . Hence

$$T_b^* \mathcal{K} \cong \mathcal{L}_\eta \otimes T_{\sigma(a)}^*(\mathcal{H} \circ \sigma^{-1}) = \mathcal{L}_\eta \otimes T_{\sigma(a)}^* \mathcal{K} \ell_{n,q}(\psi, \chi \circ \sigma^{-1}) \mid \mathbb{G}_{m/\overline{\mathbb{F}}_p}.$$

By [Fis92, Lemma 4.11], we have $b = \sigma(a)$ and $\rho = (\chi \circ \sigma^{-1})\eta$ as unordered tuples. This implies that

$$\text{Kl}_n(\psi, \rho, q, b) = \eta(b) \text{Kl}_n(\psi, \chi, q, a).$$

Hence both Kloosterman sums vanish or $\eta(b) = \lambda^{-1}$. $\square$

Remark 2.8. In [Fis92, Corollary 4.27], Fisher showed that if $p > (2n^{4d} + 1)^2$ and

$$|\text{Kl}_n(\psi, \chi, q, a)| = |\text{Kl}_n(\psi, \rho, q, b)|,$$

then $b = \sigma(a)$, $\rho = (\chi \circ \sigma^{-1})\eta$, or $b = (-1)^n \sigma(a)$, $\rho = (\chi^{-1} \circ \sigma^{-1})\eta$.

Corollary 2.9. *Keeping the hypotheses of [Theorem 2.7](#). Assume that χ is defined over $\mathbb{F}_p$, that's to say, $\chi = \chi_0 \circ \mathbf{N}_{\mathbb{F}_q/\mathbb{F}_p}$ for some n -tuple χ_0 of characters on $\mathbb{F}_p^\times$. If*

$$\mathrm{Kl}_n(\psi, \chi, q, a) = \lambda \mathrm{Kl}_n(\psi, \chi, q, b), \quad \lambda \in \mu_{q-1},$$

then $b = \sigma(a)$ for some $\sigma \in \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$, and $\mathrm{Kl}_n(\psi, \chi, q, a) = \mathrm{Kl}_n(\psi, \chi, q, b)$.

Proof. In this case, we have $\chi = \chi\eta$ and then $\eta = 1$. The result then follows easily. $\square$

3. THE NON-VANISHING OF KLOOSTERMAN SUMS

The case $n = 1$ is trivial. We will assume that $n \geq 2$ in this section.

Theorem 3.1. *Assume that $p > (3n-1)c - n$ and for any i, j , $\chi_i = \chi_j$ if $\chi_i^n = \chi_j^n$. Then $\mathrm{Kl}_n(\psi, \chi, q, a)$ is nonzero.*

Proof. Let $\mathfrak{p}$ be a prime above p in $\mathbb{Q}(\mu_{q-1})$ and $\mathfrak{P}$ the unique prime above $\mathfrak{p}$ in $\mathbb{Q}(\mu_{(q-1)p})$. Let v be the normalized $\mathfrak{P}$ -adic valuation. Once we fix an isomorphism from $\mathbb{F}_q$ to the residue field of $\mathbb{Q}(\mu_{q-1})$ at $\mathfrak{p}$, the Teichmüller lifting of the residue map at $\mathfrak{p}$ gives a primitive character ω of $\mathbb{F}_q^\times$. Denote by

$$g(m) := \sum_{t \in \mathbb{F}_q^\times} \omega^{-m}(t) \psi(\mathrm{Tr}(t))$$

the *Gauss sum*. Then the Stickelberger's congruence theorem tells that

$$v(g(m)) = \sum_{j=0}^{d-1} m_j, \tag{3.1}$$

where

$$0 \leq m \leq q-2, \quad m = \sum_{j=0}^{d-1} m_j p^j, \quad 0 \leq m_j \leq p-1,$$

see [\[Sti90\]](#) or [\[Was97, Chapter 6\]](#).

For each $i \in \{1, 2, \dots, n\}$, there is s_i such that $\chi_i = \omega^{-s_i}$. Take $x = x_1 \cdots x_n a^{-1}$ in the identity

$$\sum_{m=0}^{q-2} \omega^{-m}(x) = \begin{cases} q-1, & \text{if } x = 1; \\ 0, & \text{if } x \neq 1, \end{cases}$$

we get

$$(q-1) \mathrm{Kl}_n(\psi, \chi, q, a) = \sum_{m=0}^{q-2} \omega^m(a) \prod_{i=1}^n g(m + s_i).$$

There is a unique m such that $v(\prod_{i=1}^n g(m + s_i))$ is minimal by [Proposition 3.2](#). This implies that $\mathrm{Kl}_n(\psi, \chi, q, a)$ is nonzero. $\square$

We may assume that $1 \leq s_i \leq q-1$ (notice the bound). Write

$$s_i = \sum_{j=0}^{d-1} s_{ij} p^j$$

with $0 \leq s_{ij} \leq p-1$.

Proposition 3.2. *Assume that $p > (3n - 1)c - n$ and for any i, j , $\chi_i = \chi_j$ if $\chi_i^n = \chi_j^n$. Then there is a unique $0 \leq m \leq q - 2$ such that $v(\prod_{i=1}^n g(m + s_i))$ is minimal.*

Proof. Since $c(\chi\chi_1^{-1}) \leq c(\chi)$, we may assume that $\chi_1 = 1, s_1 = q - 1$ for simplicity.

Step 1: express the valuation in terms of m_{ij} and s_{ij} .

For each $i \in \{1, 2, \dots, n\}$, let $\epsilon_{i,-1} \in \{0, 1\}$ be the integer part of $(m + s_i)/(q - 1)$. Then we may write

$$m + s_i - (q - 1)\epsilon_{i,-1} = \sum_{j=0}^{d-1} m_{ij}p^j, \quad 0 \leq m_{ij} \leq p - 1.$$

By the Stickelberger's congruence theorem [Eq. \(3.1\)](#), we have

$$v\left(\prod_{i=1}^n g(m + s_i)\right) = \sum_{i=1}^n \sum_{j=0}^{d-1} m_{ij}. \quad (3.2)$$

Note that

$$m + s_i - (q - 1)\epsilon_{i,-1} = \sum_{j=0}^{d-1} (m_j + s_{ij})p^j - (q - 1)\epsilon_{i,-1}.$$

For each $j \in \{0, 1, 2, \dots, d - 1\}$, denote by ϵ_{ij} the integer part of $(m_j + s_{ij} + \epsilon_{i,j-1})/p$ inductively. Then $\epsilon_{ij} \in \{0, 1\}$ and

$$\begin{aligned} & \sum_{j=0}^{d-1} (m_j + s_{ij} + \epsilon_{i,j-1} - p\epsilon_{ij})p^j \\ &= m + s_i + \epsilon_{i,-1} - q\epsilon_{i,d-1} \\ &= \sum_{j=0}^{d-1} m_{ij}p^j + q(\epsilon_{i,-1} - \epsilon_{i,d-1}). \end{aligned}$$

Since both the left-hand side and $\sum_{j=0}^{d-1} m_{ij}p^j$ lie in the interval $[0, q - 2]$, we have

$$m_{ij} = m_j + s_{ij} + \epsilon_{i,j-1} - p\epsilon_{ij}$$

and $\epsilon_{i,d-1} = \epsilon_{i,-1}$.

Step 2: express the valuation in terms of $s_{\sigma_j(u_j),j}$.

There exists a permutation $\sigma_j \in S_n$ such that

$$s_{\sigma_j(1),j} \geq s_{\sigma_j(2),j} \geq \dots \geq s_{\sigma_j(n),j}. \quad (3.3)$$

If $s_{ij} = s_{i'j}$, then by [Lemma 3.3](#), $\chi_i^n = \chi_{i'}^n$, $\chi_i = \chi_{i'}$ and $\epsilon_{ij} = \epsilon_{i'j}$. If $s_{ij} > s_{i'j}$, then

$$s_{ij} + \epsilon_{i,j-1} \geq s_{i'j} + \epsilon_{i',j-1} \quad \text{and} \quad \epsilon_{ij} \geq \epsilon_{i'j}.$$

In other words, $\{\epsilon_{ij}\}_i$ and $\{s_{ij} + \epsilon_{i,j-1}\}_i$ have the same orderings as [Eq. \(3.3\)](#). Therefore, there exists $0 \leq u_j \leq n$ such that

$$\epsilon_{\sigma_j(1),j} = \dots = \epsilon_{\sigma_j(u_j),j} = 1, \quad \epsilon_{\sigma_j(u_j+1),j} = \dots = \epsilon_{\sigma_j(n),j} = 0.$$

This implies that

$$m_{\sigma_j(1),j} \geq \dots \geq m_{\sigma_j(u_j),j}, \quad m_{\sigma_j(u_j+1),j} \geq \dots \geq m_{\sigma_j(n),j}.$$

Note that $s_1 = q - 1, s_{1j} = p - 1$ and $\epsilon_{1,-1} = 1$. One can show that $\epsilon_{1,j} = 1$ inductively, which means $u_j \neq 0$. If $u_j \neq n$ and $m_{\sigma_j(u_j),j} \geq m_{\sigma_j(n),j}$, then

$$0 \geq s_{\sigma_j(u_j),j} + \epsilon_{\sigma_j(u_j),j} - p \geq s_{\sigma_j(n),j} + \epsilon_{\sigma_j(n),j} \geq 0,$$

which forces that

$$s_{\sigma_j(u_j),j} = p - 1, \epsilon_{\sigma_j(u_j),j} = 1, \quad s_{\sigma_j(n),j} = \epsilon_{\sigma_j(n),j} = 0.$$

By [Lemma 3.3](#), this implies that $\chi_{\sigma_j(u_j)}^n = \chi_{\sigma_j(n)}^n$. Then $\chi_{\sigma_j(u_j)} = \chi_{\sigma_j(n)}$ and $\epsilon_{\sigma_j(u_j),j} = \epsilon_{\sigma_j(n),j}$, which is impossible. Hence

$$m'_j := m_{\sigma_j(u_j),j} = m_j + s_{\sigma_j(u_j),j} + \epsilon_{\sigma_j(u_j),j-1} - p$$

is the unique minimum among $\{m_{1j}, m_{2j}, \dots, m_{nj}\}$. Therefore, the valuation [Eq. \(3.2\)](#) becomes

$$\begin{aligned} \sum_{i,j} m_{ij} &= \sum_{i,j} (m_j + s_{ij} + \epsilon_{i,j-1} - p\epsilon_{ij}) \\ &= \sum_{i,j} (m'_j - s_{\sigma_j(u_j),j} - \epsilon_{\sigma_j(u_j),j-1} + p + s_{ij} + \epsilon_{i,j-1} - p\epsilon_{ij}) \\ &= ndp + \sum_j \left(\sum_i s_{ij} + n(m'_j - s_{\sigma_j(u_j),j} - \epsilon_{\sigma_j(u_j),j-1}) + u_{j-1} - pu_j \right) \\ &= ndp + \sum_{i,j} s_{ij} + n \sum_j \left(m'_j - s_{\sigma_j(u_j),j} - \frac{p-1}{n}u_j - \epsilon_{\sigma_j(u_j),j-1} \right). \end{aligned} \quad (3.4)$$

Here, $u_{-1} := \sum_{i=1}^d \epsilon_{i,-1} = u_{d-1}$.

Step 3: find the minimal valuation.

If

$$\left| \left(s_{\sigma_j(i),j} + \frac{p-1}{n}i \right) - \left(s_{\sigma_j(i'),j} + \frac{p-1}{n}i' \right) \right| \leq 1,$$

then by [Lemma 3.3](#), $\chi_{\sigma_j(i)}^n = \chi_{\sigma_j(i')}^n$, $\chi_{\sigma_j(i)} = \chi_{\sigma_j(i')}$, $s_{\sigma_j(i),j} = s_{\sigma_j(i'),j}$. This implies that $i = i'$ because $(p-1)/n > 1$. Therefore, there exists a unique U_j such that

$$s_{\sigma_j(U_j),j} + \frac{p-1}{n}U_j = \max_{1 \leq i \leq n} \left\{ s_{\sigma_j(i),j} + \frac{p-1}{n}i \right\}.$$

Moreover,

$$s_{\sigma_j(U_j),j} + \frac{p-1}{n}U_j > s_{\sigma_j(i),j} + \frac{p-1}{n}i + 1 \quad (3.5)$$

for any $i \neq U_j$.

Write

$$E_{\sigma_j(1),j} = \dots = E_{\sigma_j(U_j),j} = 1, \quad E_{\sigma_j(U_j+1),j} = \dots = E_{\sigma_j(n),j} = 0.$$

If m is

$$M = \sum_{j=0}^{d-1} M_j p^j, \quad \text{where } M_j = p - s_{\sigma_j(U_j),j} - E_{\sigma_j(U_j),j-1},$$

then $m'_j = 0, \epsilon_{ij} = E_{ij}$ and $u_j = U_j$. Denote by V the corresponding valuation [Eq. \(3.2\)](#) for $m = M$.

If all $u_j = U_j$, then $\epsilon_{ij} = E_{ij}$ and

$$\sum_{i,j} m_{ij} = V + n \sum_j m'_j \geq V.$$

The equality holds if and only if all $m'_j = 0$, i.e., $m = M$. If there exists j such that $u_j \neq U_j$, then by Eqs. (3.4) and (3.5), we have

$$\begin{aligned}
& \frac{1}{n} \left(\sum_{i,j} m_{ij} - V \right) \\
&= \sum_j \left(m'_j - s_{\sigma_j(u_j),j} - \frac{p-1}{n} u_j - \epsilon_{\sigma_j(u_j),j-1} \right) \\
&\quad - \sum_j \left(-s_{\sigma_j(U_j),j} - \frac{p-1}{n} U_j - E_{\sigma_j(U_j),j-1} \right) \\
&\geq \sum_j \left(s_{\sigma_j(U_j),j} + \frac{p-1}{n} U_j - s_{\sigma_j(u_j),j} - \frac{p-1}{n} u_j + E_{\sigma_j(U_j),j-1} - \epsilon_{\sigma_j(u_j),j-1} \right) \\
&\geq \sum_{u_j \neq U_j} \left(s_{\sigma_j(U_j),j} + \frac{p-1}{n} U_j - s_{\sigma_j(u_j),j} - \frac{p-1}{n} u_j - 1 \right) > 0.
\end{aligned}$$

Hence the valuation Eq. (3.2) is minimal if and only if $m = M$. $\square$

Lemma 3.3. Assume that $p > (3n-1)c - n$. If $\chi_i^n \neq \chi_{i'}^n$, then for each j , there is no integer $0 \leq \alpha \leq n$ such that $|s_{ij} - s_{i'j} - \frac{p-1}{n}\alpha| \leq 1$.

Proof. There exist integers r, r' such that

$$s_i = \frac{(q-1)r}{c}, \quad s_{i'} = \frac{(q-1)r'}{c}.$$

Then

$$s_{ij} = \frac{a_{j+1}p - a_j}{c}, \quad s_{i'j} = \frac{a'_{j+1}p - a'_j}{c},$$

where $a_j \equiv rp^{-j}$, $a'_j \equiv r'p^{-j} \pmod{c}$ with $1 \leq a_j, a'_j \leq c$. Let $a''_j := a_j - a'_j$. Then $|a''_j| \leq c-1$.

If

$$\frac{p-1}{n}\alpha + t = s_{ij} - s_{i'j} = \frac{a''_{j+1}p - a''_j}{c}$$

for an integer $0 \leq \alpha \leq n$ and a real number t with $|t| \leq 1$, then

$$(na''_{j+1} - \alpha c)p = na''_j - \alpha c + nct.$$

There are three cases:

- If $na''_{j+1} - \alpha c \neq 0$ and $\alpha = n$, then $a''_{j+1} \neq c$,
 $p \leq |(a''_{j+1} - c)p| = |a''_j - c + ct| \leq 3c - 1 \leq (3n-1)c - n$
since $n \geq 2$.
- If $na''_{j+1} - \alpha c \neq 0$ and $\alpha < n$, then
 $p \leq |na''_j - \alpha c + nct| \leq n(c-1) + c(n-1) + nc \leq (3n-1)c - n$.
- If $na''_{j+1} - \alpha c = 0$, then $n(r-r') \equiv n(a_{j+1} - a'_{j+1})p^{j+1} = \alpha cp^{j+1} \equiv 0 \pmod{c}$.
That is to say, $\chi_i^n = \chi_{i'}^n$.

These finish the proof. $\square$

Remark 3.4. When $n = 2$, $p > 3c-2$ is enough by a careful estimation. See [Zha21, Lemma 3.4, Proposition 3.6].

4. PROOF OF THE MAIN THEOREM

Theorem 4.1. *Assume that $p > \max \{(2n^{2d} + 1)^2, (3n - 1)c - n\}$ and for any i, j , $\chi_i = \chi_j$ if $\chi_i^n = \chi_j^n$. Then $\text{Kl}_n(\psi, \chi, q, a)$ generates $\mathbb{Q}(\mu_{pc})^H$, where H consists of those $\sigma_t \tau_w$ such that there exists an integer k and a character η satisfying*

$$t^n = a^{1-p^k}, \quad \chi^w = \chi^{p^k} \eta, \quad \eta(a) = \prod \chi^w(t).$$

Proof. Note that if χ is Kummer-induced, then there is a non-trivial character Λ such that $\chi = \chi \Lambda$ and $\Lambda^n = 1$. Thus there exists $i \neq j$ such that $\chi_i = \chi_j \Lambda$ and $\chi_i^n = \chi_j^n$, which contradicts to our assumptions. Certainly, $\chi = (\xi_1, \xi_1^{-1}, 1, \Lambda_2) \xi_2$ is also impossible.

By Theorems 2.7 and 3.1, and the fact that

$$\sigma_t \tau_w \text{Kl}_n(\psi, \chi, q, a) = \prod \chi^{-w}(t) \text{Kl}_n(\psi, \chi^w, q, at^n),$$

we have that $\sigma_t \tau_w$ fixes $\text{Kl}_n(\psi, \chi, q, a)$ if and only if

$$at^n = \sigma(a), \quad \chi^w = (\chi \circ \sigma^{-1})\eta, \quad \eta(\sigma(a)) = \prod \chi^w(t)$$

for some $\sigma \in \text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ and character η . Write $\sigma(x) = x^{p^{-k}}$. Since $t^p = t$, we have

$$t^n = t^{np^k} = (\sigma(a)/a)^{p^k} = a^{1-p^k},$$

$$\eta(a) = \eta(\sigma(a))^{p^k} = \prod \chi^w(t^{p^k}) = \prod \chi^w(t)$$

and $\chi^w = \chi^{p^k} \eta$. □

Remark 4.2. Denote by $\alpha = \gcd(k, d)$ and $\lambda := a^{p^\alpha - 1}$. Since the order of a divides $\gcd((p^k - 1)(p - 1), p^d - 1) = (p^\alpha - 1) \gcd(p - 1, \frac{p^d - 1}{p^\alpha - 1}) = (p^\alpha - 1) \gcd(p - 1, \frac{d}{\alpha})$,

we have $\lambda^{d/\alpha} = 1$. If $\lambda \neq 1$, then

$$\text{Tr}(a) = (1 + \lambda + \cdots + \lambda^{\frac{d}{\alpha} - 1}) \cdot (a + a^p + \cdots + a^{p^{\alpha-1}}) = 0.$$

Hence, $\text{Tr}(a) \neq 0$ implies that $\lambda = 1, t^n = a^{1-p^k} = 1$. If moreover $\chi = \mathbf{1}$, then

$$H = \{t \in \text{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q}) \mid t^n = 1\}.$$

In fact, this holds for any p , see [Wan95]. See also [KRV11] for an attempt on a weaker condition.

Remark 4.3. Consider the Kloosterman sums

$$S_m = \text{Kl}(\psi, \chi \circ \mathbf{N}_{\mathbb{F}_{q^m}/\mathbb{F}_q}, q^m, a).$$

The L -function

$$L(T) = \exp \left(\sum_{m=1}^{\infty} \frac{T^m}{m} S_m \right)$$

is a rational function over $\mathbb{Q}(\mu_{p(q-1)})$ by the Dwork-Bombieri-Grothendick rationality theorem. Thus the sequence $\{S_m\}_m$ is a linear recurrence sequence. As shown in [WY20, Theorem 3], the sequence $\{\mathbb{Q}(S_m)\}_{m \geq N}$ is periodic of period r for some r, N .

Assume that for any i, j , $\chi_i = \chi_j$ if $\chi_i^n = \chi_j^n$. By [Theorem 1.1](#), if $p > \max \{(2n^{2dm} + 1)^2, (3n - 1)c - n\}$, then $\mathbb{Q}(S_m) = \mathbb{Q}(\mu_{pc})^H$, where H consists of those $\sigma_t \tau_w$ such that there exists an integer k and a character η on $\mathbb{F}_q^\times$ satisfying

$$t^n = a^{1-p^k}, \quad \chi^w = \chi^{p^k} \eta, \quad \eta(a) = \gamma \cdot \prod \chi^w(t) \text{ with } \gamma^m = 1. \quad (4.1)$$

Hence $\mathbb{Q}(S_m) = \mathbb{Q}(S_{m-c})$ since $\gamma^c = 1$.

If $p > \max \{(2n^{2d(N+r)} + 1)^2, (3n - 1)c - n\}$, then the generating field of S_m is determined by [Eq. \(4.1\)](#) for any m . But unfortunately, we do not have a bound on N . We roughly guess that S_m has the predicted generating field if $p > 3ndc$.

5. EXAMPLES

Denote by $n_0 := (n, p - 1)$, d_0 the degree of $a^{(1-p)/n_0}$ and

$$a_0 := \mathbf{N}_{\mathbb{F}_{p^{d_0}}/\mathbb{F}_p} \left(a^{(1-p)/n_0} \right) = a^{(1-p^{d_0})/n_0}.$$

Since

$$(a^{(1-p)/n_0})^{p^k - 1} = t^{(p-1)n/n_0} = 1,$$

we have $k = d_0\beta$ for some integer β . Moreover,

$$t^n = a^{1-p^k} = a_0^{n_0(1-p^k)/(1-p^{d_0})} = a_0^{n_0\beta}.$$

5.1. The case $n = 2$.

Proposition 5.1. *Let $\chi = \{1, \chi\}$, where χ is a multiplicative character of order $c \neq 2$. If $p > \max \{(2^{2d+1} + 1)^2, 5c - 2\}$, then $\text{Kl}(\psi, \chi, p^d, a)$ generates $\mathbb{Q}(\mu_{pc})^H$, where*

$$H = \begin{cases} \langle \tau_{q_0} \sigma_{a_0}, \sigma_{-1}, \tau_{-1} \rangle, & \text{if } \chi(-1) = 1, \chi(a) = 1; \\ \langle \tau_{-q_0} \sigma_{a_0}, \sigma_{-1} \rangle, & \text{if } \chi(-1) = 1, \chi(a) = \chi(a_0) = -1; \\ \langle \tau_{q_0^\alpha} \sigma_{a_0^\alpha}, \sigma_{-1} \rangle, & \text{if } \chi(-1) = 1, \chi(a)^\alpha \neq 1; \\ \langle \tau_{q_0} \sigma_{-a_0}, \tau_{-1} \sigma_{-1} \rangle, & \text{if } \chi(-1) = -1, \chi(a) = \chi(a_0) = -1; \\ \langle \tau_{q_0} \sigma_{a_0}, \tau_{-1} \rangle, & \text{if } \chi(-1) = -1, \chi(a) = 1; \\ \langle \tau_{q_0} \sigma_{a_0}, \tau_{-1} \sigma_{-1} \rangle, & \text{if } \chi(-1) = -1, \chi(a) = -1, \chi(a_0) = 1; \\ \langle \tau_{q_0^{\alpha/2}} \sigma_{-a_0^{\alpha/2}} \rangle, & \text{if } \chi(-1) = -1, 2 \mid \alpha, \chi(a) \neq \pm 1; \\ \langle \tau_{q_0^\alpha} \sigma_{a_0^\alpha} \rangle, & \text{if } \chi(-1) = -1, 2 \nmid \alpha, \chi(a) \neq \pm 1. \end{cases}$$

is a subgroup of $\text{Gal}(\mathbb{Q}(\mu_{pc})/\mathbb{Q})$, $q_0 = \#\mathbb{F}_p(a^{(1-p)/2})$, $a_0 = a^{(1-q_0)/2} \in \mathbb{F}_p^\times$ and α is the order of $\chi(a_0) \in \mu_{p-1}$.

Proof. As remarked above, $k = d_0\beta$ and $t^2 = a_0^{2\beta}$ for some integer β , where $q_0 = p^{d_0}$. Hence $t = \pm a_0^\beta$ and

$$\chi^w = \{1, \chi^w\} = \chi^{q_0^\beta} \eta = \left\{ \eta, \eta \chi^{q_0^\beta} \right\}, \quad \eta(a) = \chi^w(t).$$

There are two cases:

(i) If $\eta = 1$, $\chi^w = \chi^{q_0^\beta}$, then $w \equiv q_0^\beta \pmod{c}$ and

$$1 = \eta(a) = \chi^w(t) = \chi(t) = \chi(\pm a_0^\beta).$$

- (ii) If $\eta = \chi^w, \eta\chi^{q_0^\beta} = 1$, then $w \equiv -q_0^\beta \pmod{c}$. Since $\chi^w(a) = \eta(a) = \chi^w(t)$, we have $\chi(a) = \chi(t) = \chi(\pm a_0^\beta)$. Since $a_0 = a^{(1-q_0)/2} \in \mathbb{F}_p^\times$, we have

$$\chi(a_0)^2 = \chi(a)^{1-q_0} = \chi(a_0)^{(1-q_0)\beta} = 1.$$

Thus $\chi(a_0) = \pm 1$ and $\alpha = 1$ or 2 .

The case $\chi(-1) = 1$.

- (i) $\beta = \alpha m$ for some m and $w \equiv q_0^{\alpha m}, t = \pm a_0^{\alpha m}$.
(ii) If $\alpha = 1$, $\chi(a_0) = \chi(a) = 1$, then $w \equiv -q_0^m, t = \pm a_0^m$; if $\alpha = 2$, $\chi(a_0) = \chi(a) = -1$, then $w \equiv -q_0^{1+2m}, t = \pm a_0^{1+2m}$.

The case $\chi(-1) = -1$ and $2 \mid \alpha$.

- (i) $w \equiv q_0^{\alpha m}, t = a_0^{\alpha m}$ or $w \equiv q_0^{\alpha(m+1/2)}, t = -a_0^{\alpha(m+1/2)}$.
(ii) $\alpha = 2$, $\chi(a) = \chi(a_0) = -1$. Then $w \equiv -q_0^{1+2m}, t = a_0^{1+2m}$ or $w \equiv -q_0^{2m}, t = -a_0^{2m}$.

The case $\chi(-1) = -1$ and $2 \nmid \alpha$.

- (i) $w \equiv q_0^{\alpha m}, t = a_0^{\alpha m}$.
(ii) $\alpha = 1$ and $\chi(a_0) = 1$. If $\chi(a) = 1$, then $w \equiv -q_0^m, t = a_0^m$; if $\chi(a) = -1$, then $w \equiv -q_0^m, t = -a_0^m$. $\square$

Example 5.2. If $a \in \mathbb{F}_p^\times$, then $q_0 = p, \alpha = 1$ or 2 . One can easily obtain that

$$H = \begin{cases} \langle \tau_p, \tau_{-1}, \sigma_{-1} \rangle, & \text{if } \chi(-1) = 1 \text{ and } \chi(a) = 1; \\ \langle \tau_p, \sigma_{-1} \rangle, & \text{if } \chi(-1) = 1 \text{ and } \chi(a) = -1; \\ \langle \tau_p, \tau_{-1} \rangle, & \text{if } \chi(-1) = -1 \text{ and } \chi(a) = 1; \\ \langle \tau_p, \tau_{-1}\sigma_{-1} \rangle, & \text{if } \chi(-1) = -1 \text{ and } \chi(a) = -1; \\ \langle \tau_p \rangle, & \text{if } \chi(-1) = -1 \text{ and } \chi(a) \neq \pm 1. \end{cases}$$

This drops the combinatorial condition on (p, d) and the non-vanishing condition on $\text{Tr}(a)$ in [Zha21, Theorems 1.1, 1.3], while we require that p is large with respect to d .

Remark 5.3. Assume that $\chi = \Lambda_2$. If $\Lambda_2(a) \neq 1$, then the Kloosterman sum vanishes. If $\Lambda_2(a) = 1$ and $\text{Tr}(\sqrt{a}) \neq 0$, then the Kloosterman sum generates $\mathbb{Q}(\mu_p)^+$ if $\chi(-1) = 1$; $\mathbb{Q}(\mu_p)$ if $\chi(-1) = -1$. See [Zha21, Theorem 1.1(1)].

5.2. The upper bound of the generating field. If $\eta = 1$, then $\chi_i^w = \chi_i^{q_0^\beta}$. Thus $w \equiv q_0^\beta \pmod{c}$. Denote by

$$\alpha := \min \{ \alpha \in \mathbb{Z}_{>0} \mid \exists t_0 \in \mathbb{F}_p^\times \text{ such that } t_0^n = a_0^{n_0\alpha}, \prod \chi(t_0) = 1 \}.$$

Write $\beta = \alpha s + r, 0 \leq r < \alpha$. Then

$$(tt_0^{-s})^n = a_0^{n_0\beta - n_0\alpha s} = a_0^{n_0r}, \quad \prod \chi(tt_0^{-s}) = 1.$$

This forces $r = 0$ and $t = \lambda t_0^s$ with $\lambda^n = 1, \prod \chi(\lambda) = 1$. Hence

$$H \supseteq H_0 := \langle \tau_{q_0^\alpha} \sigma_{t_0}, \sigma_\lambda \mid \lambda^n = 1, \prod \chi(\lambda) = 1 \rangle$$

and $\text{Kl}(\psi, \chi, q, a) \in \mathbb{Q}(\mu_{pc})^{H_0}$. This gives an upper bound of the degree of $\text{Kl}(\psi, \chi, q, a)$.

Example 5.4. Denote by $m(\xi)$ the multiplicity of ξ in the n -tuple χ . Assume that there exists a character ξ such that $m(\xi) \neq m(\xi')$ for any $\xi' \neq \xi$. Then one can easily show that $\eta = 1$ and $H = H_0$.

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